

Algebras of Information A New and Extended Axiomatic Foundation

Prof. Dr. Jürg Kohlas,

Dept. of Informatics DIUF
University of Fribourg
CH – 1700 Fribourg (Switzerland)
E-mail: juerg.kohlas@unifr.ch
<http://diuf.unifr.ch/tcs>

Version: January 11, 2017

Contents

1	Introduction	5
I	Labeled Algebras	13
2	Conditional Independence	15
2.1	Quasi-Separoids	15
2.2	Arithmetic of Partitions	18
2.3	Families of Compatible Frames	22
3	Labeled Information Algebras	29
3.1	Axiomatics	29
3.2	Valuation Algebras	35
3.3	Semiring Information Algebras	42
4	Local Computation	55
4.1	Markov Trees	55
4.2	Computing in Markov Trees	62
4.3	Computation in Hypertrees	66
II	Domain-Free Algebras	71
5	Domain-Free Information Algebras	73
5.1	Unlabeling of Information	73
5.2	Domain-Free Axiomatics	75
5.3	Duality	81
6	Order of Information	87
6.1	The Idempotent Case	87
6.2	Regular Algebras	88
6.3	Separative Algebras	93

7	Proper Information	99
7.1	Ideal Completion	99
7.2	Compact Algebras	102
7.3	Duality For Compact Algebras	114
7.4	Continuous Algebras	124
7.5	Atomic Algebras	134
III	Constructing New Algebras	143
8	Information Maps	145
8.1	Continuous Maps	145
8.2	Cartesian Closed Categories	151
9	Random Maps	155
9.1	Simple Random Variables	155
9.2	Random Mappings	162
9.3	Random Variables	168
10	Allocations of Probability	179
10.1	Algebra of Allocations of Probability	179
10.2	Random Mappings and Allocations	187
11	Support Functions	199
11.1	Characterisation	199
11.2	Generating Support Functions	206
11.3	Canonical Random Mappings	210
11.4	Minimal Extensions	219
11.5	The Boolean Case	225
	References	235

Chapter 1

Introduction

The basic idea behind information algebras (Kohlas, 2003a; Kohlas & Schmid, 2014) is that information comes in pieces, each referring to a certain question, that these pieces can be combined or aggregated and that the part relating to a given question can be extracted. This algebraic structure can be given different forms. Questions are often represented by a lattice of domains, and a popular model is based on the subset lattice of a set of variables. Pieces of information are then represented by valuations associated with these domains. This leads then to an algebraic structure called valuation algebras (Kohlas, 2003a). The axiomatics of this algebraic structure was in essence proposed by (Shenoy & Shafer, 1990a). Valuation algebras have already many important applications in Computer Science related to constraint systems, relational databases, different uncertainty formalisms like probability, belief functions, fuzzy set and possibility measures, and many more, we refer to (Pouly & Kohlas, 2011). An important particular case of valuation algebras, both from practical as well as theoretical point of views, are *idempotent* valuation algebras, also called proper information algebras: The combination of a piece of information with itself or part of itself gives nothing new. This allows to introduce an order between pieces of information reflecting information content. It relates proper information algebras also to domain theory (Kohlas, 2003a; Kohlas & Schmid, 2014).

The basic view of information as pieces which can be combined, which relate to questions and from which the part relating to given questions can be extracted, leads to two different but essentially equivalent algebraic structure, *labeled* and *domain-free* valuation algebras (Kohlas, 2003a; Kohlas & Schmid, 2014). The original proposal of an axiomatics in (Shenoy & Shafer, 1990a) was in labeled form; later (Shafer, 1991) proposed the domain-free form. However, for valuation algebras, the two forms are not fully equivalent, there are labeled forms which have no domain-free form and vice versa. An important contribution of this paper is to give a new axiomatic system, where there exists a full duality between these two forms.

The representation of questions by lattice of domains or even subsets of variables is unnecessarily restrictive and excludes important applications in Computer Science. Already early work on belief functions (Shafer, 1976) considered a reference structure for belief functions called *family of compatible frames*. This is not covered by valuation algebras. In this paper a much more general abstract framework for representing questions is proposed and based on it a new system of axioms for information algebras, covering the previous forms of valuation algebras and proper information algebras as special cases. Originally, the theory of valuation algebras in (Shenoy & Shafer, 1990a) was motivated by the desire to generalise the local computation scheme for probabilities proposed in (Lauritzen & Spiegelhalter, 1988) for other formalisms of uncertainty, especially belief functions. This goal will also be maintained for the new algebraic structures presented here. We claim however, that these algebraic structures represent moreover essential features of information in general, beyond particular uncertainty calculi. In probability theory, conditional independence structures between variables are essential for efficient local computation. It has been known since long that structures of conditional independence can be generalised beyond probability (Studeny, 1993; Shenoy, 1994b; Studeny, 1995). In fact, we claim that conditional independence is a basic issue for information and information algebras in general. In (Dawid, 2001) a fundamental mathematical structure called *separoids* is abstracted underlying all the concepts of conditional independence and its applications. It is shown in this paper that an even weaker concept (called here *quasi-separoid*) is sufficient to allow for local computation schemes in the context of information algebras in appropriate conditional independence structures.

The basis of the theory of information algebras as developed here, is the relation of *conditional independence* among questions or domains representing them. In Chap. 2 it is argued that questions should be partially ordered according to their granularity, their acuteness or coarseness of the possible answers. In fact, this partial order is required in the present theory to form a join-semilattice. The join of two questions is the coarsest among all questions finer than both original questions; the join represents thus the *combined* question of the two original ones. In addition, a three-place relation among questions is required which describes the conditional independence of two questions, given a third one. This relation is requested to satisfy four conditions, which are natural requirements for a concept of conditional independence. In fact a separoid, the usual concept for modelling conditional independence and irrelevance, satisfies (among others) these conditions. Therefore, a join-semilattice together with a three-place relation satisfying these conditions is called a *quasi-separoid* (or q-separoid). An important source of q-separoids are join-semilattices of partitions of some universe and they form useful models of systems of questions. Somewhat more general than partitions are *families of compatible frames* (f.c.f). This

notion has been introduced in (Shafer, 1976). Here a slightly modified version of this concept is proposed and it is shown how q-separoids arise from f.c.f. Both q-separoids or partitions of f.c.f generalise the most often used multivariate model, where questions are represented by families of variables and their domains, as in Bayesian networks, belief functions, etc. In this last case, q-separoids become separoids and this links our general theory to the more classical approach to valuation and information algebras.

Q-separoids model questions. In Chap 3, pieces of information are added, each piece referring to an element of the q-separoid, to a determined question. But information can to be transported or extracted relative to other questions, and also pieces of information can be combined or aggregated. The corresponding operations are introduced and the required properties of them are stated as axioms. In particular, the operations of transport and combination are related to conditional independence. This determines then a *labeled information algebra*. For certain particular q-separoids, the axioms can be transformed into those of classical valuation algebras, (Shenoy & Shafer, 1990a; Kohlas, 2003a). The latter appear in this way as particular cases of the general information algebras treated in this text. In relation to partition and f.c.f q-separoids, pieces of information may be represented by subsets of the universe or of frames. These set information algebras are important models of information algebras.

A general problem of information processing can be formulated in the framework of information algebras as combining a number of pieces of information and then extracting from the combination the part corresponding to one or several given questions. Formulated in this way, this may well be computationally infeasible. For probabilistic networks (Lauritzen & Spiegelhalter, 1988) proposed a computational scheme which avoids this problem by organising the computations in such a way that combination and extraction always can be done on the small domains of the pieces of information involved in the combination. This is called *local computation*. In (Shenoy & Shafer, 1990a) it was shown that local computation can be applied much more generally to valuation algebras. Here we demonstrate that it can be used in the even more general framework of information algebras. The underlying structures of the domains of the information to be used in local computation is called in the literature join or junction trees, hypertrees or Markov trees. These are concepts which are defined relative to multivariate models. They determine certain structures of conditional independence. These structures exist also with respect to the much more general concept of q-separoids. That implies that local computation can be executed in Markov trees as defined relative to q-separoids. However, whereas the concepts of join trees, hypertrees and Markov trees are equivalent in the particular case of multivariate models, in the sense that one notion may be transformed into another one and vice versa, this is no more true in the case of general q-separoids. It turns out that the basic conditional independence structure for local computation in

information algebras is the one of *Markov trees*. As labeled algebras, they are based on q-separoids.

Labeled information algebras are convenient for computations. But there is an alternative form of information algebras, namely domain-free information algebras. They are introduced in part II. These algebras are more adapted for theoretical, algebraic considerations. Domain-free information algebras are derived from labeled ones by unlabeled (Chap. 5). As in a labeled algebra, there is the operation of combination or aggregation of information. The operation of transport of a piece of information from its domain to another domain becomes an operation of *information extraction*, by which the part of a piece of information relevant to a given domain is extracted. Conversely, labeled information algebras may be obtained from domain-free ones. This establishes a full duality between these two forms; they are the two sides of a coin.

Some pieces of information may be more informative than others. This is reflected by some order between the elements of an information algebra: A piece of information is more informative than a second one, if it is obtained from the latter by combination with a third one. In the important case of *idempotent* information algebras, this establishes a natural *partial order* in the information algebra which respects combination and extraction: Combined information of two or more pieces of information is more informative than each of its parts. Also, by extraction, information can only be lost. This partial order of information in idempotent information algebras is very important and is studied and exploited in later parts of the text. However, even in the general, non idempotent case, the relation is of interest and important. It determines no more a partial, but only a *preorder* in the information algebra. This preorder however is no more natural in general, in particular, extraction does not respect the order in all cases. Such preorders are also studied in semigroup theory and there *regular* semigroups are of particular interest. This notion can be extended to valuation algebras (Kohlas, 2003a) and even to information algebras as understood here. And it turns out that in regular information algebras the preorder becomes natural. Even more general are *separative* information algebras, and in this framework the preorder is still natural. These questions of information order are discussed in Chap. 6.

For the rest of part II and also part III only *idempotent* information algebras are considered. As mentioned above, they can be considered as “proper” information algebras: The combination of a piece of information with part of it does not give new information. Mathematically speaking, the partial information order mentioned above adds an essential new element, which is exploited in Chap. 7. In the partial information order of proper information algebras, combination generates the supremum of the pieces of information combined. In this way the pieces of information determine a join-semilattice. Consistent collections of pieces of information form then

an ideal in this semilattice. These ideals form themselves an information algebra, extending and completing the original one to a complete lattice. Furthermore, in computation only finite pieces of information can be processed, whereas general pieces of information may be approximated by finite ones. This idea can be expressed by the well-known order-theoretic notion of finite or compact elements and this idea is also the motivation of domain theory as a theory of computation (Gierz, 2003). In this respect, proper information algebras become particular instances of domains. Like in this theory, *algebraic* and *continuous* information algebras can be defined and considered. The essential points which distinguishes the theory of proper information algebra from domain theory, is the presence of the extraction operators and that approximation of information takes place not only globally in the algebra but locally on each domain of the underlying q-separoid. The discussion refers to domain-free information algebras, but the extension of duality covering labeled algebras is also addressed.

Most constructions of universal algebra to obtain new algebras from existing ones, apply to information algebras. But in part III we restrict ourselves to constructions which make sense from the point of view of information. Again, we limit in this part ourselves to idempotent information algebras. So, in Chap. 8 we consider information maps, that is, maps, which take elements from a first information algebra as input and map them to elements of a second information algebra as output. Such maps should be monotone: more information as input should result in more information as output. Such maps represent themselves information and in fact, they form an information algebra. If the input comes from a continuous algebra, the information maps should be continuous, that is respect convergence in some sense. It turns out that continuous maps form a continuous information algebra. Both, general idempotent information algebras together with monotone maps and algebraic or continuous information algebras together with continuous maps, determine Cartesian closed categories.

Information in practice is often uncertain. Whether a piece of information is valid, may depend whether some assumptions are satisfied or not. This can be modelled by mappings from a space of assumptions to a proper information algebra. Some assumptions may be more likely to hold, more probable. So, it makes sense to assume the space of assumptions to be a probability space (Chap. 9). In this view, Chap. 9 becomes an abstract theory of probabilistic argumentation, generalising (Haenni *et al.*, 2000; Kohlas, 2003b). It is also a generalisation of the theory of hints (Kohlas & Monney, 1995), and its application to statistical inference (Kohlas & Monney, 2007). Such random mappings form again information algebras, and according to various restrictions imposed on these maps different algebras may be obtained.

Random maps may be used to evaluate hypotheses according to their likelihood. For this purpose, the probability of the assumptions supporting

a given hypothesis may be considered. At first, this poses some problems of measurability. However, following (Shafer, 1973; Shafer, 1979), from the probability space a probability algebra may be obtained and a so-called allocation of probability may be derived. This allows to associate a numerical degree of support to any hypothesis which can be formulated within the information algebra or its ideal completion. Our interpretation of this notion by probabilistic argumentation differs from Shafer's epistemological interpretation as partial belief; the mathematics however is the same. What is new, is that it is shown that idempotent information algebras provide the natural, general framework for such a theory, much more general than the classical set-based theory. In fact, allocations of probability may be defined independent of random mappings. And they represent again information, they can be given the structure of idempotent information algebras (Chap. 10). If the allocations are derived from random mappings, the algebra of allocations is homomorphic to the one of random mappings, in some cases even isomorphic. However in the most general case, a random mapping carries more information than its associated allocation of probability.

The question arises, whether any allocation of probability in an information algebra may be derived from a random mapping. This question is addressed in Chap. 11 in the context of support functions. This is an extension of the studies in (Shafer, 1973; Kohlas & Monney, 1994). Again it is shown that the natural mathematical context for this study are information algebras and the question is answered in the positive. All these three last chapters are in fact an extension of what is usually called Dempster-Shafer theory of evidence and discussed with respect to fields of sets, instead of information algebras.

It must be emphasised that there are many subjects and issues regarding information algebras, not addressed here. We hint at a few of important subjects and issues.

Local computation is based on Markov trees associated with combinations of pieces of information. How to find appropriate Markov trees for a given combination? In the multivariate setting, join trees (which in this case are also Markov trees) can be found by selecting a sequence of variable eliminations. An extensive literature presents heuristics for "good" elimination sequences. All this does not carry over to q-separoids in general, not even to partition- or f.c.f-separoids and nothing is known so far about finding Markov trees in these general cases. So, this is a big open question, which is of course basic for the practical application of local computation beyond multivariate models.

Also, there are several architectures for local computation for probabilistic networks, which make use of some partial division (or information elimination). These architectures apply also for general valuation algebras in the multivariate setting, if some appropriate concept of information elimination is assumed (Shenoy, 1994a). Mathematically speaking, what is re-

quired for these computational schemes are regular or separative (labeled) valuation algebras (Kohlas, 2003a). Now, there exist in fact regular and separative (dual) labeled versions of the corresponding domain-free information algebras discussed in Chap. 6. Instances of such labeled valuation algebras are semiring-valued algebras, where the semiring is regular or separative (Kohlas & Wilson, 2006). This again, is something that applies also to generalised information algebras. Further we remark that regular and separative valuation algebras provide the natural frame for a rigorous mathematical theory of compositional modelling (Jirousek, 1997; Jirousek, 2011; Jirousek & Shenoy, 2014; Jirousek & Shenoy, 2015).

Set algebras are natural examples of generalised information algebras; the elements of sets represent precise answers to the questions represented by set domains, like in partitions or families of compatible frames. How are abstract information algebras related to set algebras? In (Kohlas, 2003a) it was shown that any labeled valuation algebra can be embedded into a relational set algebra. This subject is considered in (Kohlas & Schmid, 2016) for domain-free valuation algebras, including in particular distributive lattice and Boolean information algebras. It is shown that the classical representation theory of Boolean algebras and distributive lattices based on the topological Stone and Priestley spaces can be extended to the corresponding information algebras. It is conceivable, that this representation theory extends also to generalised information algebras. It is also known that topology plays an important role in domain theory and here too an extension of these topological approaches to idempotent information algebras, especially algebraic and continuous ones could be envisaged.

Finally, there are many examples and instances of valuation algebras known in the multivariate setting, see for instance (Pouly & Kohlas, 2011). Much less examples are studied in the present more general context. Of course, set algebras, belief functions and probabilistic argumentation and a few others are presented in this text. These are examples generalised from the multivariate setting. The question arises whether other examples from this setting which make sense in the more general setting and whether there are other genuine examples in the non-multivariate framework of generalised information algebras.

Part I

Labeled Algebras

Chapter 2

Conditional Independence

2.1 Quasi-Separoids

Information informs about possible answers to questions. Therefore, the first task in developing an algebraic theory of information consists in modelling appropriate systems of questions. At this place we do not try to describe the internal structure of questions, we rather think of questions as represented by certain abstract domains describing or representing somehow the possible answers to questions; for instance, in the simplest case, by listing the possible answers. This idea will be pursued in Sections 2.2 and 2.3. But here, at this point, we want to be more general. Let D be a set whose elements are thought to represent domains. Its generic elements will be denoted by lowercase letters like x, y, z, \dots . We assume that domains or questions can be compared with respect to their granularity or fineness. Therefore we require $(D; \leq)$ to be a *partial order*, where $x \leq y$ means that y is finer than x , that is, answers to y will be more informative than answers to x . Moreover, if x and y are two elements of D , we want to be able to consider the *combined question* represented by x and y . This is surely a question finer than both x and y . So, the combined question should be the coarsest question finer than x and y , that is the *supremum* $\sup\{x, y\}$ or *join* $x \vee y$ of x and y . Therefore we assume that D contains with any pair x, y also its join $x \vee y$. Hence we require that (D, \leq) is a *join-semilattice*. In most discussions of valuation algebras so far, $(D; \leq)$ has been assumed to be a lattice, even a distributive one (for instance the lattice of subsets of variables), see (Shenoy & Shafer, 1990a; Shafer & Shenoy, 1990; Kohlas, 2003a). Only recently in (Kohlas & Schmid, 2014; Kohlas & Schmid, 2016) this requirement has been relaxed.

Besides order, representing granularity of questions, a further relation between questions is needed which describes *conditional independence* of two questions, given a third one. Therefore, in D a relation $x \perp y | z$ is assumed which is thought to express the idea that an information relative to x , does

restrict the possible answers to y only through its part relative to z , and vice versa. Or, in other words, only the part relative to z of an information relative to x is relevant as an information relative to y , and vice versa. Rather than to give some explicit definition of this relation in D , we only require it to satisfy the following four conditions:

- C1** $x \perp y | y$ or all $x, y \in D$,
- C2** $x \perp y | z$ implies $y \perp x | z$,
- C3** $x \perp y | z$ and $w \leq y$ imply $x \perp w | z$,
- C4** $x \perp y | z$ implies $x \perp y \vee z | z$.

A join-semilattice D together with a relation $x \perp y | z$, $(D; \leq, \perp)$, satisfying conditions C1 to C4 will be called a *quasi-separoid* (or also *q-separoid*). In Sections 2.2 and 2.3 explicit forms of such relations will be given. The meaning of the relation will further be made operational in terms of operations on pieces of information in Chapter 3. In Chapter 4 the role of conditional independence in local computation schemes will be clarified. So far, in valuation algebras, the concept of conditional independence was implicit in the axiomatic system rather than explicit (Kohlas, 2003a). However for example in (Kohlas & Monney, 1995) it became important in an explicit form for local computation. The system presented here is abstracted from there.

In the literature two additional conditions are assumed for a relation of conditional independence (Dawid, 2001):

- C5** $x \perp y | z$ and $w \leq y$ imply $x \perp y | z \vee w$,
- C6** $x \perp y | z$ and $x \perp w | y \vee z$ imply $x \perp y \vee w | z$.

Then D is called a *separoid*. If $(D; \leq)$ is a lattice, then yet another condition can be added:

- C7** If $z \leq y$ and $w \leq y$, then $x \perp y | z$ and $x \perp y | w$ imply $x \perp y | z \wedge w$.

With this additional condition D is called a *strong separoid*. For a detailed discussion of separoids we refer to (Dawid, 2001). For example it can be shown that C1 to C3 together with C5 and C6 imply C4. For our purposes, that is in particular for the study of local computation (Section 4), C1 to C4 are sufficient. This is thought to be one of the main results of this text.

Partitions and families of compatible frames, as studied in Sections 2.2 and 2.3, provide important examples of quasi-separoids, where D is in general only a join-semilattice. Here, we discuss briefly the case where D is a lattice, as for example the lattice of subsets of variables. Define $x \perp_L y | z$ to hold if and only if

$$(x \vee z) \wedge (y \vee z) = z. \quad (2.1)$$

Theorem 2.1 *If $(D; \leq)$ is a lattice, then the relation $x \perp_L y|z$ defines a quasi-separoid.*

Proof. We have $(x \vee y) \wedge (y \vee y) = y$, hence C1 is satisfied. By the symmetry of the definition, C2 holds too. If $w \leq y$, then $z \leq (x \vee z) \wedge (w \vee z) \leq (x \vee z) \wedge (y \vee z) = z$, so C3 follows. Finally from (2.1) we see that C4 is valid. \square

If $x \leq y$, then from $x \perp_L y|y$ (C1) it follows that $x \perp_L x|y$ by C3. Now, in some cases $x \perp_L x|y$ implies $x \leq y$. A separoid with this property is called basic, see (Dawid, 2001). We adapt this to call a quasi-separoid *basic*, if $x \perp_L x|y$ implies $x \leq y$. The following theorem was proved in (Dawid, 2001) for a basic separoid, but it is valid for basic quasi-separoids too.

Theorem 2.2 *Suppose $(D; \leq)$ is a lattice. Then a q -separoid $(D; \leq, \perp)$ is basic if and only if*

$$x \perp_L y|z \Rightarrow (x \vee z) \wedge (y \vee z) = z. \quad (2.2)$$

Proof. If (2.2) holds, then $x \perp_L x|y$ implies $x \vee y = y$, hence $x \leq y$.

Suppose now that $x \perp_L y|z$. Then $(x \vee z) \perp_L (y \vee z)|z$ by C4 and C2. Define $w = (x \vee z) \wedge (y \vee z)$ such that $w \leq x \vee z$ and $w \leq y \vee z$. Using C3 and C2 we deduce then that $w \perp_L w|z$. So, if the quasi-separoid is basic, we obtain that $w \leq z$. Since we always have $w \geq z$, it follows that $w = z$. \square

If we meet both sides of (2.1) with x we obtain $x \wedge (y \vee z) = x \wedge z$ which is equivalent to

$$x \wedge (y \vee z) \leq z. \quad (2.3)$$

This condition is equivalent to (2.1) if the lattice D is *modular*. So, in this case we have $x \perp_L y|z$ if and only if (2.3) holds.

Theorem 2.3 *If $(D; \leq)$ is a lattice, the relation $x \perp_L y|z$ defines a separoid if and only if $(D; \leq)$ is modular.*

Proof. Assume D modular. We are going to show that C5 and C6 are satisfied. If D is modular, then $x \wedge (y \vee z) = x \wedge z$ if and only if $x \perp_L y|z$. So, if $w \leq y$, it follows $x \wedge (y \vee z \vee w) = x \wedge (y \vee z)$. Therefore, $x \wedge (z \vee w) \leq x \wedge (y \vee z \vee w) = x \wedge (y \vee z) = x \wedge z \leq x \wedge (z \vee w)$, hence $x \wedge (y \vee (z \vee w)) = x \wedge (z \vee w)$. This shows that $x \perp_L y|z \vee w$, that is C5. Further, $x \perp_L y|z$ and $x \perp_L w|y \vee z$ imply $x \wedge (y \vee z) = x \wedge z$ and $x \wedge (w \vee y \vee z) = x \wedge (y \vee z)$. Together, this leads to $x \wedge (w \vee y \vee z) = x \wedge z$, hence $x \perp_L (y \vee w)|z$. So C6 holds.

On the other hand, assume $x \perp_L y|z$ to be a separoid. By (2.1) we have $x \perp_L y|x \wedge y$. Thus, if $z \leq x$, by C5, it follows that $x \perp_L y|(x \wedge y) \vee z$. This means that $x \wedge (y \vee z) = (x \wedge y) \vee z$, which is modularity. \square

Further, (2.1) implies that

$$x \wedge y \leq z. \quad (2.4)$$

If the lattice D is *distributive*, then $(x \vee z) \wedge (y \vee z) = (x \wedge y) \vee z$. In this case (2.1) is equivalent to (2.4).

Theorem 2.4 *If $(D; \leq)$ is a distributive lattice the relation $x \perp_L y | z$ defines a strong separoid.*

Proof. A distributive lattice is modular, so C5 and C6 hold. It remains to prove C7. Assume D distributive so that $x \perp_L y | z$ if and only if (2.4). Now $x \perp_L y | z$ and $x \perp_L y | w$ imply $x \wedge y \leq z$ and $x \wedge y \leq w$, hence $x \wedge y \leq z \wedge w$, which shows that $x \perp_L y | z \wedge w$. Therefore C7 is satisfied. \square

We may also consider the relation $x \perp_{dy} | z$ which holds if and only if $x \wedge y \leq z$. The following theorem is due to (Dawid, 2001):

Theorem 2.5 *The relation $x \perp_{dy} | z$ is a separoid if and only if $(D; \leq)$ is a distributive lattice.*

In a distributive lattice $x \perp_L y | z$ if and only if $x \perp_{dy} | z$ by the discussion above. Therefore if $x \perp_{dy} | z$ is a separoid, it is a strong separoid by Theorem 2.4.

An important instance of a distributive lattice is the lattice of the subsets of a set I . If s, t, r denote subsets of I , then $s \perp_L t | r$ if and only if $s \cap t \subseteq r$. This is then a strong separoid by the theorems above. This is the classical case considered in the large majority of studies on conditional independence. We shall come back to this case and its background in the next section and later.

2.2 Arithmetic of Partitions

An important concept for representing questions in the sense of the previous section is given by partitions of some universe U representing a set of possible worlds. Questions x can be modelled by equivalence relations \equiv_x on U , the idea being that we have $u \equiv_x u'$ if question x has the same answer in the worlds u respectively u' . Equivalence relations \equiv_x correspond bijectively to partitions P_x of U . The members of such partitions will be called *blocks*, the blocks B of the partition P_x are the equivalence classes $B = \{u \in U : u \equiv_x v\}$ for an arbitrary $v \in B$. A question x will be considered finer than a question y if $u \equiv_x u'$ implies $u \equiv_y u'$, or equivalently, if every block of P_x is contained in a (unique) block of P_y . This implies that every block of P_y is the union of blocks of P_x . We carry this over to partitions and write in this case $P_y \leq P_x$. This is a partial order among the partitions of U , in fact it is the opposite order of $(\text{Part}(U), \leq)^\theta$ usually considered in the literature. Under this (opposite) order, the join of two partitions P_1 and P_2 of U , written $P_1 \vee P_2$, is given by the partition whose blocks are the nonempty intersections $B_1 \cap B_2$ of blocks B_1 of P_1 and B_2 of P_2 .

$(Part(U), \leq)$ is in fact a lattice (Grätzer, 1978), so there exists also a meet between finite sets of partitions. We come back to this later.

We are going to define a relation of *conditional independence* between partitions. For a finite set of partitions P_1, \dots, P_n , $n \geq 2$ we define

$$R(P_1, \dots, P_n) = \{(B_1, \dots, B_n) : B_i \in P_i, \cap_{i=1}^n B_i \neq \emptyset\}.$$

So, R contains the tuples of mutually compatible blocks, representing compatible answers to the questions represented by P_1, \dots, P_n . We call the partitions P_1, \dots, P_n *independent*, if $R(P_1, \dots, P_n)$ is the Cartesian product of P_1, \dots, P_n ,

$$R(P_1, \dots, P_n) = P_1 \times \dots \times P_n.$$

If P is another partitions and B a block of P , then we define (for $n \geq 1$)

$$R_B(P_1, \dots, P_n) = \{(B_1, \dots, B_n) : B_i \in P_i, \cap_{i=1}^n B_i \cap B \neq \emptyset\}.$$

This represents the tuples of all blocks of P_1, \dots, P_n compatible with $B \in P$. We call P_1, \dots, P_n *conditionally independent given P* , if for all blocks B of P ,

$$R_B(P_1, \dots, P_n) = R_B(P_1) \times \dots \times R_B(P_n).$$

In this case we write $\perp\{P_1, \dots, P_n\}|P$ or (for $n = 2$) $P_1 \perp P_2 | P$. Note that this relation holds, if $B_i \cap B \neq \emptyset$ for $i = 1, \dots, n$ it follows that $B_1 \cap \dots \cap B_n \cap B \neq \emptyset$. Such relations have been studied in (Shafer, 1976; Kohlas & Monney, 1995). We are going to relate this relation to q-separoids.

Let then D be a subset of $Part(U)$, such that (D, \leq) (in the order defined above) is a join-subsemilattice of the lattice $(Part(U), \leq)$. This gives us a q-separoid.

Theorem 2.6 *If $(D; \leq)$ is a join-subsemilattice of $(Part(U); \leq)$, then $(D; \leq, \perp)$ is a q-separoid.*

Proof. C1) and C2) are obvious. To prove C3) assume $P_1 \perp P_2 | P$ and $P_3 \leq P_2$. Assume then $B_1 \cap B \neq \emptyset$ and $B_3 \cap B \neq \emptyset$ where B_1, B_3 and B are blocks of P_1, P_3 and P respectively. Then there is a block B_2 of P_2 such that $B_2 \subseteq B_3$ and $B_2 \cap B \neq \emptyset$. But then $B_1 \cap B_3 \cap B \supseteq B_1 \cap B_2 \cap B \neq \emptyset$, hence $P_1 \perp P_3 | P$. Finally, for C4) assume $P_1 \perp P_2 | P$ and consider blocks B of P , B_1 of P_1 and C of $P_2 \vee P$ such that $B_1 \cap B \neq \emptyset$ and $C \cap B \neq \emptyset$. Then $C = B_2 \cap B$ for some block B_2 of P_2 , and then $B_2 \cap B \neq \emptyset$. It follows $B_1 \cap C \cap B = B_1 \cap B_2 \cap B \neq \emptyset$, hence $P_1 \perp P_2 \vee P | P$. \square

So, we have here a first model of a q-separoid, and in particular $(Part(U); \leq, \perp)$ is also a q-separoid. It turns out that these q-separoids are basic.

Theorem 2.7 *If $(D; \leq)$ is a join-subsemilattice of $(\text{Part}(U); \leq)$, then $(D; \leq, \perp)$ is a basic q -separoid*

Proof. Assume $P_1 \perp P_1 | P$ for two partitions P_1 and P . Then $B_1 \cap B \neq \emptyset$ and $B'_1 \cap B \neq \emptyset$ imply $B_1 \cap B'_1 \cap B \neq \emptyset$ where B_1 and B'_1 are blocks of P_1 , B a block P . But B_1 and B'_1 are either equal or disjoint, so B must be a subset of some block B_1 of P_1 , so that $P_1 \leq P$. \square

We consider now the case that (D, \leq) is a sublattice, of $(\text{Part}(U), \leq)$. Then from Theorems 2.2 and 2.7 it follows that $P_1 \perp P_2 | P$ implies $(P_1 \vee P) \wedge (P_2 \vee P) = P$. Is it possible that $P_1 \perp P_2 | P$ and $(P_1 \vee P) \wedge (P_2 \vee P) = P$ are equivalent, and if yes, under what conditions does this equivalence hold? In order to address this question, we introduce saturation operators σ_P , associated with a partition P as mappings $\mathbb{P}(U) \rightarrow \mathbb{P}(U)$, where $\mathbb{P}(U)$ denotes the power set of U , defined by

$$\sigma_P(X) = \bigcup \{B \in P : B \cap X \neq \emptyset\}$$

for subsets X of U . So, $\sigma_P(X)$ is the smallest union of blocks of P covering X . Note that P may be recovered from σ_P as the set of blocks $\{\sigma(\{x\}) : x \in U\}$. This concept will also be used later. Remark that $\sigma_P(\sigma_P(X)) = \sigma_P(X)$ and $\sigma_P(\sigma_P(X) \cap \sigma_P(Y)) = \sigma_P(X) \cap \sigma_P(Y)$. The following properties of saturation operators will be crucial for our purposes:

Lemma 2.1 *Let σ_P , $P \in \text{Part}(U)$, be a saturation operator on U . Then for all $X, Y \subseteq U$*

1. $\sigma_P(\emptyset) = \emptyset$,
2. $X \subseteq \sigma_P(X)$,
3. $X \subseteq Y$ implies $\sigma_P(X) \subseteq \sigma_P(Y)$,
4. $\sigma_P(\sigma_P(X) \cap Y) = \sigma_P(X) \cap \sigma_P(Y)$.

Proof. Items 1. to 3. are obvious. For 4. observe that $\sigma_P(X) \cap Y \subseteq \sigma_P(X) \cap \sigma_P(Y)$, so $\sigma_P(\sigma_P(X) \cap Y) \subseteq \sigma_P(X) \cap \sigma_P(Y)$ by 3. Now $\sigma_P(X) \cap \sigma_P(Y)$ is the union of all $B \in P$ satisfying $B \cap X \neq \emptyset \neq B \cap Y$. Obviously, for each such B we have $B \cap \sigma_P(X) = B$, so $B \cap \sigma_P(X) \cap Y \neq \emptyset$ and B participates in the union of all $B' \in P$ forming $\sigma_P(\sigma_P(X) \cap Y)$. \square

Given two partitions P_1 and P_2 with associated saturation operators σ_1 and σ_2 define $\sigma : \mathbb{P}(U) \rightarrow \mathbb{P}(U)$ by

$$\sigma(X) = \bigcup_{k \in \omega} \sigma_1 \circ \sigma_2 \circ \sigma_1 \circ \sigma_2 \circ \dots \circ \sigma_k(X)$$

for all $X \subseteq U$, where $\sigma_k = \sigma_1$ if k is odd and $\sigma_k = \sigma_2$ if k is even. Clearly, $\sigma(X)$ is the smallest set containing X which is a union of P_1 -blocks as well as a union of P_2 -blocks. It follows that $\sigma = \sigma_{P_1 \wedge P_2}$.

If $\sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_1$, then, since saturation operators are idempotent, $\sigma = \sigma_1 \circ \sigma_2$. In this case we say that partitions P_1 and P_2 *commute*. This turns out to be exactly the necessary and sufficient condition for $P_1 \perp P_2 | P \Leftrightarrow (P_1 \vee P) \wedge (P_2 \vee P) = P$. Note that P_1 and P_2 commute iff for any block C of $P_1 \wedge P_2$ and $B_1, B_2 \subseteq C$, where B_1 and B_2 are blocks of P_1 and P_2 respectively, it follows that $B_1 \cap B_2 \neq \emptyset$.

Theorem 2.8 *Let $P_1 \perp P_2 | P$ be the conditional independence relation defined above in a sublattice $(D; \leq)$ of $(\text{Part}(U), \leq)$. Then $P_1 \perp P_2 | P \Leftrightarrow (P_1 \vee P) \wedge (P_2 \vee P) = P$ if and only if the partitions in D pairwise commute.*

Proof. Recall that $P_1 \perp P_2 | P$ implies $(P_1 \vee P) \wedge (P_2 \vee P) = P$. Assume first that $(P_1 \vee P) \wedge (P_2 \vee P) = P$ implies $P_1 \perp P_2 | P$, so that, in particular, $P_1 \perp P_2 | P_1 \wedge P_2$. Hence, if C , B_1 and B_2 are blocks of partitions P , P_1 and P_2 respectively, then $B_1 \cap C \neq \emptyset$ and $B_2 \cap C \neq \emptyset$ imply $B_1 \cap B_2 \cap C \neq \emptyset$. But, since $P_1 \wedge P_2 \leq P_1, P_2$ we have $B_1, B_2 \subseteq C$, hence $B_1 \cap B_2 \neq \emptyset$. But this implies that the partitions P_1 and P_2 commute.

Conversely, assume that the partitions P_1 and P_2 commute and assume that $(P_1 \vee P) \wedge (P_2 \vee P) = P$. Consider blocks B_1, B_2 and B of P_1, P_2 and P respectively, and $B'_1 = B_1 \cap B \neq \emptyset$, $B'_2 = B_2 \cap B \neq \emptyset$, such that B'_1 and B'_2 are blocks of $P_1 \vee P$ and $P_2 \vee P$, and both subsets of B . Then, if the partitions commute, $B'_1 \cap B'_2 \neq \emptyset$, hence $B_1 \cap B_2 \cap B \neq \emptyset$, and so indeed $P_1 \perp P_2 | P$. \square

As in the previous section, we write $P_1 \perp_L P_2 | P$ iff $(P_1 \vee P) \wedge (P_2 \vee P) = P$. So, if the partitions of the sublattice $(D; \leq)$ of $(\text{Part}(U); \leq)$ pairwise commute, we have $P_1 \perp P_2 | P = P_1 \perp_L P_2 | P$. Recall that $P_1 \perp_L P_2 | P$ is a separoid if and only if the lattice $(D; \leq)$ is modular and if the lattice is distributive it is a strong separoid, and then $P_1 \perp_L P_2 | P$ if and only if $P_1 \perp_d P_2 | P$, that is $P_1 \wedge P_2 \leq P$. We now give an important example of a distributive sublattice of $(\text{Part}(U), \leq)$.

In practical applications, often a set of variables is considered, and the information one is interested in concerns the values of certain groups of variables. So consider a countable family of variables $X = \{X_i : i \in \mathbb{N}\}$, and let V_i the set of possible values of the variable X_i . Let

$$V_\omega = \prod_{i \in \mathbb{N}} V_i.$$

Let s be any subset of \mathbb{N} . Define for any sequence $t \in V_\omega$ its restriction to s , denoted by $t|s$, as follows: If $s = \{i_1, \dots, i_n\}$, then $t|s = (t_{i_1}, \dots, t_{i_n})$. Also, define an equivalence relation \equiv_s in V_ω by

$$t \equiv_s t' \text{ iff } t|s = t'|s. \quad (2.5)$$

Each equivalence relation \equiv_s defines a partition P_s of V_ω and $P_s \leq P_r$ iff $s \subseteq r$. The associated saturation operators σ_s produce the *cylindrical* sets

$\sigma_s(X)$ over s in V_ω and they commute. Let $D = \{P_s : s \subseteq \mathbb{N}\}$ and (D, \leq) the corresponding lattice, order-isomorph to the subset-lattice of \mathbb{N} . This lattice is *distributive*, so we have here a distributive sublattice $(D; \leq)$ of $(\text{Part}(V_\omega); \leq)$ of pairwise commuting partitions.

This is called a *multivariate model*. We usually work with the subset lattice of \mathbb{N} rather than with the associated lattice of partitions D . Then we write $s \perp t | r$ instead of $P_s \perp P_t | P_r$, which holds if $(s \cup r) \cap (t \cup r) = r$ or, equivalently $s \cap t \subseteq r$. Note that in this case, we may consider any subset u of \mathbb{N} and then the subsets of u define still a distributive lattice of pairwise commuting partitions, but this time, of partitions of V_u . And we still have by $s \perp t | r$ a q-separoid (even a separoid) in the corresponding lattice. The idea that the universe may not be really fixed, but may vary, will be pursued in the following section for an alternative explicit model of a system of questions.

2.3 Families of Compatible Frames

Here, another view of a system of domains, modelling questions, is presented. In a partition, any block can be thought of representing a possible answer to the questions represented by the partition. So, to emphasise this point of view, we may consider the set of blocks of a partition P , say $\Theta_P = \{\theta_B : B \in P\}$ as the set of possible answers to represent the question. If partition P_1 is coarser than partition P_2 , that is, if $P_1 \leq P_2$, we have a mapping τ of Θ_{P_1} into $\mathbb{P}(\Theta_{P_2})$ defined by

$$\tau(\theta_B) = \{\theta_C : C \subseteq B\}.$$

Clearly, the $\tau(\theta_B)$, for $B \in P_1$ form a partition of Θ_{P_2} . By the map τ any possible answer θ_B in Θ_{P_1} is split up into a set $\tau(\theta_B)$ of finer answers in the set Θ_{P_2} . Therefore, the map τ is called a *refining* and Θ_{P_2} a refinement of Θ_{P_1} , and conversely, Θ_{P_1} is called a coarsening of Θ_{P_2} . The following properties of refinings between frames associated with partitions, are easily verified: (1) If $P_1 \leq P_2 \leq P_3$ and τ_1 is the refining of P_1 to P_2 , τ_2 the refining of P_2 to P_3 , then $\tau_2 \circ \tau_1$ is the refining of P_1 to P_3 . (2) Since $P \leq P$, the identity map id is a refining of P to P . (3) Whenever P_1 and P_2 are two partitions, then there exists at most one refining between them, if $P_1 \leq P_2$ the refining of P_1 or if $P_2 \leq P_1$ the refining of P_2 . (4) If $P_1, P'_1 \leq P_2$ and τ, τ' are the corresponding refinings, then if for all $B \in P_1$ there exists a $B' \in P'_1$ such that $\tau(\theta_B) = \tau'(\theta_{B'})$ and for all $B' \in P'_1$ there exists a $B \in P_1$ such that $\tau'(\theta_{B'}) = \tau(\theta_B)$, then $P_1 = P'_1$. (5) If τ_1 and τ_2 are the refinings of Θ_{P_1} and Θ_{P_2} respectively to $\Theta_{P_1 \vee P_2}$, then for every $\theta_B \in \Theta_{P_1 \vee P_2}$, there exists a $\theta_{B_1} \in \Theta_{P_1}$ and a $\theta_{B_2} \in \Theta_{P_2}$, such that $\tau_1(\theta_{B_1}) \cap \tau_2(\theta_{B_2}) = \{\theta_B\}$.

These consideration motivate the definition of *families of compatible frames* (f.c.f) as follows: A *frame* Θ is simply a non-empty set, whose ele-

ments are thought to represent possible answers to the question represented by the frame. Another frame Λ may be obtained from Θ by splitting some or all elements θ of Θ . Mathematically this is represented by specifying for each $\theta \in \Theta$ the subset $\tau(\theta)$ in Λ consisting of the possibilities into which θ has been split. So, $\tau : \Theta \rightarrow \mathbb{P}(\Lambda)$ is a mapping from Θ into the power set of Λ . It must satisfy the following conditions:

1. $\tau(\theta) \neq \emptyset$ for all $\theta \in \Theta$,
2. $\tau(\theta') \cap \tau(\theta'') = \emptyset$ if $\theta' \neq \theta''$,
3. $\cup_{\theta \in \Theta} \tau(\theta) = \Lambda$.

The map τ is called a *refining* of Θ , the set Λ a *refinement* of Θ and the latter a *coarsening* of the former. Note that a refining τ determines a partition of Λ with blocks $\tau(\theta)$ for $\theta \in \Theta$. Further, a refining τ may be extended to a map of subsets of Θ into the power set of Λ :

$$\tau(S) = \bigcup_{\theta \in S} \tau(\theta),$$

for any subset S of Θ . A refining τ satisfies the following conditions, see (Shafer, 1976):

1. τ is one-to-one,
2. $\tau(S) = \emptyset$ if and only if $S = \emptyset$,
3. $\tau(\Theta) = \Lambda$,
4. $\tau(\cup\{S \in \mathcal{S}\}) = \cup\{\tau(S) : S \in \mathcal{S}\}$,
5. $\tau(\cap\{S \in \mathcal{S}\}) = \cap\{\tau(S) : S \in \mathcal{S}\}$,
6. $\tau(S^c) = \tau(S)^c$,
7. $\tau(S) \subseteq \tau(R)$ iff $S \subseteq R$.

Further, if τ_1 is a refining of Θ_1 into Θ_2 and τ_2 a refining of Θ_2 into Θ_3 , the the composition $\tau_1 \circ \tau_2$ is a refining of Θ_1 into Θ_3 .

If Λ is a refinement of Θ and τ the corresponding refining, then we define a map $v : \mathbb{P}(\Lambda) \rightarrow \mathbb{P}(\Theta)$ by

$$v(S) = \{\theta \in \Theta : \tau(\theta) \cap S \neq \emptyset\}, \quad (2.6)$$

The map v is called a *saturation operator*, due to the similarity to saturation operators for partitions.

Let \mathcal{F} be a nonempty collection of sets, called *frames*, where no pair of frames has an element in common. Further, let \mathcal{R} be a nonempty set of refining between pairs of frames of \mathcal{F} . Then we formally define:

Definition 2.1 *Family of Compatible Frames: A pair $(\mathcal{F}, \mathcal{R})$ of frames and refinings \mathcal{R} are called a family of compatible frames (f.c.f) provided the following conditions are satisfied:*

1. *Composition of Refinings: If $\tau_1 : \mathbb{P}(\Theta_1) \rightarrow \mathbb{P}(\Theta_2)$ and $\tau_2 : \mathbb{P}(\Theta_2) \rightarrow \mathbb{P}(\Theta_3)$ belong to \mathcal{R} , then $\tau_1 \circ \tau_2 \in \mathcal{R}$.*
2. *Identity: If $\Theta \in \mathcal{F}$, then the identity map $id : \mathbb{P}(\Theta) \rightarrow \mathbb{P}(\Theta)$ belongs to \mathcal{R} .*
3. *Identity of Refinings: If $\tau_1 : \mathbb{P}(\Theta) \rightarrow \mathbb{P}(\Lambda)$ and $\tau_2 : \mathbb{P}(\Theta) \rightarrow \mathbb{P}(\Lambda)$ are elements of \mathcal{R} , then $\tau_1 = \tau_2$.*
4. *Identity of Coarsenings: If $\tau_1 : \mathbb{P}(\Theta_1) \rightarrow \mathbb{P}(\Lambda)$ and $\tau_2 : \mathbb{P}(\Theta_2) \rightarrow \mathbb{P}(\Lambda)$ belong to \mathcal{R} and if for each $\theta_2 \in \Theta_2$ there exists a $\theta_1 \in \Theta_1$ and for each $\theta_1 \in \Theta_1$ there exists a $\theta_2 \in \Theta_2$ such that $\tau_1(\theta_1) = \tau_2(\theta_2)$, then $\Theta_1 = \Theta_2$.*
5. *Existence of Minimal Common Refinement: For any finite family $\Theta_1, \dots, \Theta_n$ of frames in \mathcal{F} , there exists a common refinement $\Lambda \in \mathcal{F}$ such that if $\Lambda' \in \mathcal{F}$ is another common refinement of $\Theta_1, \dots, \Theta_n$, then Λ' is also a refinement of Λ , and, if τ_i are the refinings of Θ_i to Λ , then for every $\lambda \in \Lambda$, there exist elements $\theta_i \in \Theta_i$ such that*

$$\tau_1(\theta_1) \cap \dots \cap \tau_n(\theta_n) = \{\lambda\}. \quad (2.7)$$

The concept of a f.c.f has been introduced in (Shafer, 1976), in a similar way. In (Shafer, 1976) additional conditions are required, in particular, that any frame has refinings in the family, excluding an ultimate refining. This excludes frames related to lattices of partitions. By the discussion above, it is evident, that the frames Θ_P associated with the partitions P of a join-subsemilattice of the partition lattice $(Part(U), \leq)$ of some universe, together with the corresponding refinings, constitute a f.c.f. On the other hand, a f.c.f needs not to include an ultimate refining, it is thus slightly more general than the f.c.f obtained from those join-subsemilattices.

In a f.c.f $(\mathcal{F}, \mathcal{R})$, a relation $\Theta \leq \Lambda$ can be defined to hold, if Λ is a refinement of Θ , hence the latter a coarsening of the former. In fact, the system $(\mathcal{F}; \leq)$ is a join-semilattice ¹.

Theorem 2.9 *If $(\mathcal{F}, \mathcal{R})$ is a family of compatible frames, then $(\mathcal{F}; \leq)$ is a join-semilattice.*

¹If Identity of Coarsenings does not hold, the order is only a preorder. According to (Dawid, 2001) a preorder is sufficient for the theory of separoids. It may be conjectured that this could also be true for the present theory of information algebras. For the sake of simplicity we renounce to develop this generalisation.

Proof. Reflexivity of \leq follows from Identity (2). Assume $\Theta \leq \Lambda$ and $\Lambda \leq \Theta$. Let τ_1 be the refining of Θ to Λ and τ_2 the refining of Λ to Θ . Then Θ_1 is a coarsening of Λ and $\tau_1 \circ \tau_2 = id_\Theta$ and also $\tau_2 \circ \tau_1 = id_\Lambda$ by the Identity of Refinings. Further, Λ is a coarsening of itself and for every $\theta \in \Theta$, we have $\tau_1(\theta) = \{\lambda\} = id_\Lambda(\lambda)$. By the Identity of Coarsenings (3), it follows that $\Theta = \Lambda$, hence \leq is antisymmetric. Transitivity of the relation \leq follows from Composition of Refinings (1). So the relation \leq is a partial order in \mathcal{F} .

Clearly, the minimal common refinement Λ of two frames Θ_1 and Θ_2 which exists by the requirement of the Existence of a Minimal Common Refinement is an upper bound of Θ_1 and Θ_2 . Suppose Λ' is another common refinement of Θ_1 and Θ_2 , then Λ' is a refinement of Λ , hence $\Lambda \leq \Lambda'$ and Λ is the least upper bound. So $(\mathcal{F}; \leq)$ is a join-semilattice \square

In the sequel we write $\Lambda = \Theta_1 \vee \dots \vee \Theta_n$ for the minimal common refinement of $\Theta_1, \dots, \Theta_n$.

We are now going to introduce a relation of conditional independence into a family of compatible frames, following the model of partitions, and show that it defines a quasi-separoid. The compatibility relation between the elements of different frames $\Theta_1, \dots, \Theta_n$ of a family of compatible frames $(\mathcal{F}, \mathcal{R})$ is defined first. Let $\Lambda = \Theta_1 \vee \dots \vee \Theta_n$ and $\tau_i : \mathbb{P}(\Theta_i) \rightarrow \Lambda$ the corresponding refinings of Θ_i for $i = 1, \dots, n$. Define

$$R(\Theta_1, \dots, \Theta_n) = \{(\theta_1, \dots, \theta_n) : \theta_i \in \Theta_i, \cap_{i=1}^n \tau_i(\theta_i) \neq \emptyset\}. \quad (2.8)$$

Thus R contains the tuples of mutually compatible elements θ_i . The frames $\Theta_1, \dots, \Theta_n$ are called *independent* if

$$R(\Theta_1, \dots, \Theta_n) = \Theta_1 \times \dots \times \Theta_n.$$

This relation has been studied in (Shafer, 1976) and (Cuzzolin, 2005).

Consider an element λ of a frame Λ in \mathcal{F} . We now look for tuples of elements $\theta_i \in \Theta_i$ which are compatible among themselves and with λ ,

$$R_\lambda(\Theta_1, \dots, \Theta_n) = \{(\theta_1, \dots, \theta_n) : (\theta_1, \dots, \theta_n, \lambda) \in R(\Theta_1, \dots, \Theta_n, \Lambda)\}.$$

We stress that Λ is in general not necessarily different from every Θ_i . The collection of frames $\Theta_1, \dots, \Theta_n$ is called *conditionally independent given Λ* , if for all $\lambda \in \Lambda$,

$$R_\lambda(\Theta_1, \dots, \Theta_n) = R_\lambda(\Theta_1) \times \dots \times R_\lambda(\Theta_n). \quad (2.9)$$

Then we write

$$\perp\{\Theta_1, \dots, \Theta_n\}|\Lambda$$

or $\Theta_1 \perp \Theta_2 | \Lambda$ in the case of $n = 2$. This means that once an answer λ in Λ is given, knowing an answer $\theta_i \in \Theta_i$, compatible with λ , does not restrict

the possible answers $\theta_j \in \Theta_j$ for $i \neq j$. This relation has been studied in (Kohlas & Monney, 1995), although in a slightly different system of families of compatible frames. It has been shown there that conditions C3 and C4 of a q-separoid are fulfilled (see (Kohlas & Monney, 1995), Theorems 7.14 and 7.17). We generalise this result to our present case.

Theorem 2.10 *Let $(\mathcal{F}, \mathcal{R})$ be a family of compatible frames. Then $(\mathcal{F}; \leq, \perp)$, where the relation of conditional independence is defined as above, is a q-separoid.*

Proof. In order to verify condition C1 of a q-separoid consider $\theta \in \Theta_2$. Then $R_\theta(\Theta_2) = \{\theta\}$ and $R_\theta(\Theta_1) = \{\theta_1 : \tau_1(\theta_1) \cap \tau_2(\theta) \neq \emptyset\}$, where τ_1 and τ_2 are the refinings of Θ_1 and Θ_2 to $\Theta_1 \vee \Theta_2$ respectively. Finally, we have

$$R_\theta(\Theta_1, \Theta_2) = \{(\theta_1, \theta) : \tau_1(\theta_1) \cap \tau_2(\theta) \neq \emptyset\}.$$

Thus $R_\theta(\Theta_1, \Theta_2) = R_\theta(\Theta_1) \times R_\theta(\Theta_2)$ and therefore $\Theta_1 \perp \Theta_2 | \Theta_2$.

Condition C2 follows from the symmetry of the definition of conditional independence.

In order to prove C3 assume $\Theta_1 \perp \Theta_2 | \Lambda$ and $\Theta'_2 \leq \Theta_2$. We must prove that $\Theta_1 \perp \Theta'_2 | \Lambda$, that is,

$$R_\lambda(\Theta_1, \Theta'_2) = R_\lambda(\Theta_1) \times R_\lambda(\Theta'_2)$$

for any $\lambda \in \Lambda$. Note that the relation on the left is always contained in the Cartesian product on the right. So we need only to show that any pair (θ_1, θ'_2) with $\theta_1 \in R_\lambda(\Theta_1)$ and $\theta'_2 \in R_\lambda(\Theta'_2)$ belongs to $R_\lambda(\Theta_1, \Theta'_2)$. Let now τ , τ_1 and τ_2 be the refinings of Λ , Θ_1 and Θ_2 to $\Theta_1 \vee \Theta_2 \vee \Lambda$ and ω the refining of Θ'_2 to Θ_2 . Then if $\theta_1 \in R_\lambda(\Theta_1)$ and $\theta'_2 \in R_\lambda(\Theta'_2)$,

$$\tau_1(\theta_1) \cap \tau(\lambda) \neq \emptyset, \quad \tau_2(\omega(\theta'_2)) \cap \tau(\lambda) \neq \emptyset.$$

We must prove that this implies

$$\tau_1(\theta_1) \cap \tau_2(\omega(\theta'_2)) \cap \tau(\lambda) \neq \emptyset, \tag{2.10}$$

because this means that $(\theta_1, \theta'_2) \in R_\lambda(\Theta_1, \Theta'_2)$. There is an element η in $\tau_2(\omega(\theta'_2)) \cap \tau(\lambda)$. But then there must be an element $\theta_2 \in \omega(\theta'_2)$ such that $\eta \in \tau_2(\theta_2) \cap \tau(\lambda)$, hence we conclude that $\tau_2(\theta_2) \cap \tau(\lambda) \neq \emptyset$. Then, $\Theta_1 \perp \Theta_2 | \Lambda$ implies

$$\tau_1(\theta_1) \cap \tau_2(\theta_2) \cap \tau(\lambda) \neq \emptyset.$$

Since $\tau_2(\theta_2) \subseteq \tau_2(\omega(\theta'_2))$ we conclude that (2.10) holds, hence $\Theta_1 \perp \Theta'_2 | \Lambda$ and C3 is valid.

In order to prove C4 assume $\Theta_1 \perp \Theta_2 | \Lambda$ and consider the refinings τ_1 of Θ_1 to $\Theta_1 \vee \Theta_2 \vee \Lambda$, τ_2 of Θ_2 to $\Theta_2 \vee \Lambda$, τ of Λ to $\Theta_2 \vee \Lambda$ and finally τ' of $\Theta_2 \vee \Lambda$

to $\Theta_1 \vee \Theta_2 \vee \Lambda$. In order to prove that $\Theta_1 \perp \Theta_2 \vee \Lambda | \Lambda$ we must show that for any pair of elements $\theta_1 \in R_\lambda(\Theta_1)$, $\theta'_2 \in R_\lambda(\Theta_2 \vee \Lambda)$ and $\lambda \in \Lambda$,

$$\tau_1(\theta_1) \cap \tau'(\theta'_2) \cap \tau'(\tau(\lambda)) \neq \emptyset \quad (2.11)$$

since this means that (θ_1, θ'_2) belongs to $R_\lambda(\Theta_1, \Theta_2 \vee \Lambda)$ and therefore

$$R_\lambda(\Theta_1, \Theta_2 \vee \Lambda) = R_\lambda(\Theta_1) \times R_\lambda(\Theta_2 \vee \Lambda).$$

By the Existence of Minimal Common Refinement there is are elements θ_2 in Θ_2 and $\lambda' \in \Lambda$ such that

$$\tau_2(\theta_2) \cap \tau(\lambda') = \{\theta'_2\}.$$

The assumptions that $\theta_1 \in R_\lambda(\Theta_1)$ and $\theta'_2 \in R_\lambda(\Theta_2 \vee \Lambda)$ imply that

$$\tau_1(\theta_1) \cap \tau'(\tau(\lambda)) \neq \emptyset, \quad \tau'(\theta'_2) \cap \tau'(\tau(\lambda)) \neq \emptyset.$$

Then we see that

$$\tau'(\theta'_2) = \tau'(\tau_2(\theta_2) \cap \tau(\lambda')) = \tau'(\tau_2(\theta_2)) \cap \tau'(\tau(\lambda')) \neq \emptyset$$

and further

$$\emptyset \neq \tau'(\theta'_2) \cap \tau'(\tau(\lambda)) = \tau'(\tau_2(\theta_2)) \cap \tau'(\tau(\lambda')) \cap \tau'(\tau(\lambda)).$$

This implies that $\lambda' = \lambda$, hence $\theta_2 \in R_\lambda(\Theta_2)$. From $\Theta_1 \perp \Theta_2 | \Lambda$, $\theta_1 \in R_\lambda(\Theta_1)$ and $\theta_2 \in R_\lambda(\Theta_2)$, we obtain that

$$\tau_1(\theta_1) \cap \tau'(\tau_2(\theta_2)) \cap \tau'(\tau(\lambda)) \neq \emptyset.$$

But then we have also

$$\tau_1(\theta_1) \cap \tau'(\tau_2(\theta_2) \cap \tau(\lambda)) \cap \tau'(\tau(\lambda)) \neq \emptyset.$$

If we replace here $\tau_2(\theta_2) \cap \tau(\lambda)$ by θ'_2 , we have (2.11). \square

We now show that the quasi-separoid of a family of compatible frames is basic, just as the q-separoid of partitions.

Theorem 2.11 *Let $(\mathcal{F}, \mathcal{R})$ be a family of compatible frames, then the q-separoid $\Theta_1 \perp \Theta_2 | \Lambda$ is basic.*

Proof. Assume $\Theta \perp \Theta | \Lambda$ and consider an element λ of Λ , and

$$R_\lambda(\Theta, \Theta) = \{(\theta_1, \theta_2) : \theta_1, \theta_2 \in \Theta, (\theta_1, \theta_2, \lambda) \in R(\Theta, \Theta, \Lambda)\}.$$

Consider the frame $\Theta \vee \Lambda$ and denote by τ, ω the refinings of Θ and Λ to their minimal common refinement $\Theta \vee \Lambda$ respectively. Then $(\theta_1, \theta_2, \lambda) \in R(\Theta, \Theta, \Lambda)$ if $\tau(\theta_1) \cap \tau(\theta_2) \cap \omega(\lambda) \neq \emptyset$. This implies $\theta_1 = \theta_2$ and we obtain

$$R_\lambda(\Theta, \Theta) = \{(\theta, \theta) : \theta \in \Theta, \tau(\theta) \cap \omega(\lambda) \neq \emptyset\}.$$

Further,

$$R_\lambda(\Theta) = \{\theta : \theta \in \Theta, (\theta, \lambda) \in R(\Theta, \Lambda)\}. \quad (2.12)$$

Here, $(\theta, \lambda) \in R(\Theta, \Lambda)$ iff $\tau(\theta) \cap \omega(\lambda) \neq \emptyset$. By the assumption $\Theta \perp \Theta | \Lambda$ we have $R_\lambda(\Theta, \Theta) = R_\lambda(\Theta) \times R_\lambda(\Theta)$. This is only possible, if both $R_\lambda(\Theta, \Theta)$ and $R_\lambda(\Theta)$ contain each only one single element (θ, θ) and θ respectively. So, to any element λ from Λ there is only one compatible element θ in Θ . Therefore, from $\tau(\theta) \cap \omega(\lambda) \neq \emptyset$ it follows that $\tau(\theta) \supseteq \omega(\lambda)$. Further, due to (2.7), $\tau(\theta) \cap \omega(\lambda) \neq \emptyset$ implies $\tau(\theta) \cap \omega(\lambda) = \{\chi\}$ for some element χ of the minimal common refinement $\Theta \vee \Lambda$. It follows that $\omega(\lambda) = \{\chi\}$. Identity of Coarsenings implies then $\Theta \vee \Lambda = \Lambda$. Thus we conclude that $\Theta \leq \Lambda$. \square

Again, a multivariate model, since it can be seen as a system of partitions, can also be seen as a particular case of a f.c.f. This gives us important q-separoids for applications.

Chapter 3

Labeled Information Algebras

3.1 Axiomatics

Consider a quasi-separoid $(D; \leq, \perp)$. We think of the elements of D as domains, representing questions and their answers and are now adding pieces of information relating to these questions. So, let Φ be a set whose generic elements will be denoted by lower case Greek letters like ϕ, ψ, \dots . These elements are thought to represent pieces of information, each relating to a specific question, that is to an element of D ; pieces of information which can be combined or aggregated and from which information can be extracted relative to different questions or domains in D . These ideas will be captured by the operations of labeling, which informs about the question a piece of information relates to, combination, which represents aggregation of two or more pieces of information, and transport, which describes the extraction of the information relating to a given question from a piece of information. Formally, we consider the following operations:

1. *Labeling*: $d : \Phi \rightarrow D, \phi \mapsto d(\phi)$.
2. *Combination*: $\cdot : \Phi \times \Phi \rightarrow \Phi, (\phi, \psi) \mapsto \phi \cdot \psi$.
3. *Transport*: $t : \Phi \times D \rightarrow \Phi, (\phi, x) \mapsto t_x(\phi)$.

These operations are required to satisfy the following axioms.

A0 *Quasi-Separoid*: $(D; \leq, \perp)$ is a quasi-separoid.

A1 *Semigroup*: $(\Phi; \cdot)$ is a commutative semigroup.

A2 *Labeling*: $d(\phi \cdot \psi) = d(\phi) \vee d(\psi)$, $d(t_x(\phi)) = x$.

A3 *Unit and Null:* For all $x \in D$ there are elements 0_x (null) and 1_x (unit) with $d(0_x) = d(1_x) = x$ and such that

1. $\phi \cdot 0_x = 0_x$ and $\phi \cdot 1_x = \phi$ if $d(\phi) = x$,
2. $t_y(\phi) = 0_y$ if and only if $\phi = 0_{d(\phi)}$,
3. $\phi \cdot 1_x = t_{d(\phi) \vee x}(\phi)$,
4. $1_x \cdot 1_y = 1_{x \vee y}$.

A4 *Transport:* if $x \perp y | z$ and $d(\phi) = x$, then

$$t_y(\phi) = t_y(t_z(\phi)). \quad (3.1)$$

A5 *Combination:* If $x \perp y | z$ and $d(\phi) = x$, $d(\psi) = y$, then

$$t_z(\phi \cdot \psi) = t_z(\phi) \cdot t_z(\psi). \quad (3.2)$$

A6 *Identity:* If $d(\phi) = x$, then $t_x(\phi) = \phi$.

Axiom A1 implies in particular associativity of combination, such that pieces of information may be combined in any order to obtain the same result. The unit 1_x represents vacuous information relative to the domain x . Combining it with any other piece of information on the same domain does not change the information. Further, vacuous information remains vacuous, when transported to any other domain (see Lemma 3.1 below). The null elements 0_x on the other hand destroy any information, see below, Lemma 3.1. They represent contradiction, if $\phi \cdot \psi = 0_x$, then ϕ and ψ must be considered as contradictory pieces of information. Also, transport can neither eliminate contradiction nor introduce it. Axioms A4 and A5 are important for local computation, see Section 4. They substantiate conditional independence: If $x \perp y | z$, then to transport a piece of information from x to y , only the part relating to z is relevant. Or to transport the combined pieces of information on x and y to z , only the extracted information to z is relevant. Further illustrations of the meaning of conditional independence and irrelevance are given below (Lemma 3.2).

What is characteristic and basic for this axiomatic system of information, is, beside combination, the assumption that information can be *transported* from one domain to another; that is extraction of the part of the information relevant to a specific question. This seems to be an essential property of information. There are variants of this axiomatic system, relevant for local computation, which do not allow this operation, in particular systems related to probability or Bayesian networks (see Section 3.2 below). Another important property of information, not yet present in the axiomatic system above, is that the combination of a piece of information with itself or with a part of it gives no new information. This is sometimes added as an additional axiom.

A7 Idempotency: If $d(\phi) = x$ and $y \leq x$, then $\phi \cdot t_y(\phi) = \phi$.

Although this seems essential for a concept of information, we do not require it in general. In fact, there are many instances qualifying for information, which do not satisfy idempotency. For instance a repetition of the same statement by different witnesses will enforce credibility of the fact expressed by the statement, is therefore new information.

We call a system $(\Phi, D; \leq, \perp, d, \cdot, t)$ satisfying the axioms A0 to A6 a (*generalised*) *information algebra*. The term *information algebra* has been used formerly in a slightly less general framework, and assumes also idempotency (Kohlas, 2003a; Kohlas & Schneuwly, 2009; Kohlas & Schmid, 2014). We shall see, that these axiomatic systems are particular cases of an information algebra in the present sense. This is also the case of another axiomatic system proposed by (Shenoy & Shafer, 1990a), later called valuation algebras (Kohlas & Shenoy, 2000; Kohlas, 2003a). So, the present axiomatic system generalises former systems of axioms and enlarges thus the field of application of local computation considerably (see Section 4).

Here we list a few elementary properties of an information algebra concerning transport.

Lemma 3.1

1. $y \leq z$ implies $t_y(\phi) = t_y(t_z(\phi))$,
2. $d(\phi) = x \leq y \leq z$ implies $t_z(\phi) = t_z(t_y(\phi))$.
3. $x \leq z$ and $d(\phi) = x$ imply $t_x(t_z(\phi)) = \phi$,
4. $d(\phi) = x$ and $d(\psi) = y$ imply $t_x(\phi \cdot \psi) = \phi \cdot t_x(\psi)$,
5. $d(\phi) = x$ and $d(\psi) = y$ imply $\phi \cdot \psi = t_{x \vee y}(\phi) \cdot t_{x \vee y}(\psi)$,
6. $x \leq z$ and $d(\phi) = x$ imply $t_z(\phi) = \phi \cdot 1_z$,
7. $d(\phi) = x$ implies $\phi \cdot 0_y = 0_{x \vee y}$,
8. $t_y(1_x) = 1_y$,

Proof. 1.) Assume $d(\phi) = x$. Then $x \perp z | z$ (C1) and $y \leq z$ imply $x \perp y | z$ (C3) and by A4 $t_y(\phi) = t_y(t_z(\phi))$.

2.) From $z \perp y | y$ (C1) it follows that $x \perp z | y$ (C2 and C3). Then, by axiom A4, $t_z(\phi) = t_z(t_y(\phi))$.

3.) $x \perp z | z$ (C1) and $x \leq z$ imply $x \perp x | z$ (C3) and by A4, A6 $\phi = t_x(\phi) = t_x(t_z(\phi))$.

4.) $x \perp y | x$ (C1 and C2) implies by A5, A6 that $t_x(\phi \cdot \psi) = t_x(\phi) \cdot t_x(\psi) = \phi \cdot t_x(\psi)$.

5.) $x \perp x \vee y | x \vee y$ (C1) and $y \leq x \vee y$ imply $x \perp y | x \vee y$ (C3) and by A2 and A5, A6, $\phi \cdot \psi = t_{x \vee y}(\phi \cdot \psi) = t_{x \vee y}(\phi) \cdot t_{x \vee y}(\psi)$.

6.) follows from A3 and 3.)

7.) We have $x \perp x \vee y | x \vee y$ (C1), hence by C3 also $x \perp y | x \vee y$. Using axioms A2, A3, A5 and A6 we obtain $\phi \cdot 0_y = t_{x \vee y}(\phi \cdot 0_y) = t_{x \vee y}(\phi) \cdot t_{x \vee y}(0_y) = t_{x \vee y}(\phi) \cdot 0_{x \vee y} = 0_{x \vee y}$.

8.) Assume first $x \leq y$. Then by A3 and 6.) above, $1_y = 1_x \cdot 1_y = t_y(1_x)$. Next assume $y \leq x$. Then, by 3.) above and the case just proved, $1_y = t_y(t_x(1_y)) = t_y(1_x)$. In the general case $x \perp y | x \vee y$ (C1, C3), hence by A4 and the two cases just proved $t_y(1_x) = t_y(t_{x \vee y}(1_x)) = t_y(1_{x \vee y}) = 1_y$. \square

We use these results in the sequel without further reference to the lemma. Here follow a few further results, illustrating the meaning of irrelevance and conditional independence.

Lemma 3.2 *Assume $x \perp y | z$.*

If $d(\psi_1) = x$ and $d(\psi_2) = y$, then

$$\begin{aligned} t_x(\psi_1 \cdot \psi_2) &= \psi_1 \cdot t_x(t_z(\psi_2)), \\ t_y(\psi_1 \cdot \psi_2) &= \psi_2 \cdot t_y(t_z(\psi_1)). \end{aligned}$$

If $d(\psi_1) = x$, $d(\psi_2) = y$ and $d(\psi_3) = z$, then

$$\begin{aligned} t_x(\psi_1 \cdot \psi_2 \cdot \psi_3) &= \psi_1 \cdot t_x((t_z(\psi_2) \cdot \psi_3)), \\ t_y(\psi_1 \cdot \psi_2 \cdot \psi_3) &= \psi_2 \cdot t_y((t_z(\psi_1) \cdot \psi_3)). \end{aligned}$$

Proof. 1.) We have $t_x(\psi_1 \cdot \psi_2) = \psi_1 \cdot t_x(\psi_2)$ (Lemma 3.1). But from $x \perp y | z$ by axiom A4, $t_x(\psi_2) = t_x(t_z(\psi_2))$, hence $t_x(\psi_1 \cdot \psi_2) = \psi_1 \cdot t_x(t_z(\psi_2))$. The second part follows by symmetry.

2.) Here we have $t_x(\psi_1 \cdot \psi_2 \cdot \psi_3) = \psi_1 \cdot t_x(\psi_2 \cdot \psi_3)$ (Lemma 3.1). From $x \perp y | z$ it follows that $x \perp y \vee z | z$ and then by A4, $t_x(\psi_2 \cdot \psi_3) = t_x(t_z(\psi_2 \cdot \psi_3))$. Further by Lemma 3.1 $t_z(\psi_2 \cdot \psi_3) = t_z(\psi_2) \cdot \psi_3$. So, $t_x(\psi_1 \cdot \psi_2 \cdot \psi_3) = \psi_1 \cdot t_x((t_z(\psi_2) \cdot \psi_3))$, \square

This Lemma shows that if x is conditionally independent from y given z , then only the part of ψ_2 belonging to z is needed to combine it with the information ψ_1 on x . For instance, if ψ_2 with domain y contains no information relative to z , that is $t_z(\psi_2) = 1_z$, then $t_x(\psi_1 \cdot \psi_2) = \psi_1$ and ψ_2 has no effect on domain x . Such results will be exploited for local computation in Chapter 4.

Here follows a lemma on idempotent information algebras, to be used later.

Lemma 3.3 *Assume $d(\phi) = x$ and $y \in D$. If Axiom A7 holds, then*

$$\phi \cdot t_y(\phi) = t_{x \vee y}(\phi). \quad (3.3)$$

Proof. By Lemma 3.1, 5.), and 1.) and axiom A7,

$$\begin{aligned} \phi \cdot t_y(\phi) &= t_{x \vee y}(\phi) \cdot t_{x \vee y}(t_y(\phi)) = t_{x \vee y}(\phi) \cdot t_y(\phi) \\ &= t_{x \vee y}(\phi) \cdot t_y(t_{x \vee y}(\phi)) = t_{x \vee y}(\phi). \end{aligned}$$

□

Next, we give an important example of a generalised information algebra.

Example 3.1 *Subset Algebra on a Family of Compatible Frames:* As an example for a generalised information algebra consider a family of compatible frames $(\mathcal{F}, \mathcal{R})$ with the associated conditional independence relation $\Theta_1 \perp \Theta_2 | \Lambda$ (Section 2.3). Subsets S of a frame $\Theta \in \mathcal{F}$ can be considered as pieces of information in the sense that they restrict the unknown answer in Θ to S . Let

$$\Phi_\Theta = \{(S, \Theta) : S \in \mathbb{P}(\Theta)\}$$

and

$$\Phi = \bigcup_{\Theta \in \mathcal{F}} \Phi_\Theta.$$

We consider the system $(\Phi, \mathcal{F}; \perp, d, \cdot, t)$, where $(\mathcal{F}; \leq, \perp)$ is the q-separoid introduced in Section 2.3. The operations d , \cdot and t are defined as follows:

1. *Labeling:* $d : \Phi \rightarrow \mathcal{F}$, $d(S, \Theta) = \Theta$.
2. *Combination:* $\cdot : \Phi \times \Phi \rightarrow \Phi$, defined for (S, Θ) and (R, Λ) by

$$(S, \Theta) \cdot (R, \Lambda) = (\tau_1(S) \cap \tau_2(R), \Theta \vee \Lambda),$$

where τ_1 and τ_2 are the refinings of Θ and Λ to the minimal common refinement $\Theta \vee \Lambda$ respectively.

3. *Transport:* $t : \Phi \times \mathcal{F} \rightarrow \Phi$, defined for $\Lambda \in \mathcal{F}$ and (S, Θ) by

$$t_\Lambda(S, \Theta) = (v(\tau(S)), \Lambda),$$

where τ is the refining of Θ to the common refinement $\Theta \vee \Lambda$ and v the saturation operator from $\Theta \vee \Lambda$ to Λ (see (2.6)).

The system $(\Phi, \mathcal{F}; \perp, d, \cdot, t)$ satisfies the axioms of a generalised information algebra, provided $(\mathcal{F}; \leq, \perp)$ is a q-separoid. This was already shown in (Kohlas & Monney, 1995). Although families of compatible frames were defined there slightly different than here, the proofs carry over. The semi-group condition (associativity of combination) is stated in Theorem 8.4, A4 is stated in Theorem 8.6 and A5 in Theorem 8.5 of (Kohlas & Monney, 1995).

The other conditions are more or less evident. In addition, this system satisfies also the idempotency axiom. In fact, as mentioned in Section 2.3, the lattice $\text{part}(U)$ of partitions of a universe U provides an example of a family of compatible frames. Subsets of blocks of partitions in $\text{part}(U)$ form then also an information algebra. That $(\Phi, \mathcal{F}; \perp, d, \cdot, t)$ is a generalised information algebra will also follow later from the fact that it is an instance of a semiring-valued information algebra (see Section 3.3). \ominus

Example 3.2 *Set Potentials and Belief Functions:* The previous example may be generalised by assigning to the subsets S of a frame Θ some nonnegative numbers $m(S)$. Such an assignment $m : \mathbb{P}(\Theta) \rightarrow \mathbb{R}^+ \cup \{0\}$ is called a set potential on frame Θ . To a set potential on frame Θ we attach the label $d(m) = \Theta$. If m_1 and m_2 are two set potentials on frames Θ and Λ , then a set potential m on frame $\Theta \vee \Lambda$ can be defined as follows: For a subset S of frame $\Theta \vee \Lambda$, let

$$m(S) = \sum \{m_1(S_1)m_2(S_2) : S_1 \subseteq \Theta, S_2 \subseteq \Lambda, t_{\Theta \vee \Lambda}(S_1) \cap t_{\Theta \vee \Lambda}(S_2) = S\}.$$

Then m is called the combination of m_1 and m_2 and we write $m = m_1 \cdot m_2$. Further, if m is a set potential on frame Θ and Λ any other frame, then we define a set potential $t_\Lambda(m)$ on frame Λ by

$$t_\Lambda(m)(S) = \sum \{m(T) : T \subseteq \Theta, t_\Lambda(T) = S\},$$

for any subset S of Λ . Let Φ_Θ denote the set of all set potentials on frame Θ and

$$\Phi = \bigcup_{\Theta \in \mathcal{F}} \Phi_\Theta.$$

The system $(\Phi, \mathcal{F}; \leq, \perp, \cdot, t)$ forms then a generalised information algebra. We refer to (Kohlas & Monney, 1995) and (Kohlas, 2003a) for a verification of axioms.

Set potentials may be transformed into two other set function,

$$b(S) = \sum_{T \subseteq S} m(T), \quad q(S) = \sum_{T \supseteq S} m(T).$$

There is a one-to-one relation between the set functions m , b and q . In fact, we have

$$m(S) = \sum_{T \subseteq S} (-1)^{|S-T|} b(T) = \sum_{T \supseteq S} (-1)^{|T-S|} q(T).$$

For a proof see (Shafer, 1976).

If $m(\emptyset) = 0$ and

$$\sum_{S \subseteq \Theta} m(S) = 1,$$

then the set potentials m is called a basic probability assignment, (Shafer, 1976) and the set function b is the associated belief function, the set function q the commonality function. This is the formalism of Dempster-Shafer Theory, which is described in detail in (Shafer, 1976). \ominus

In the next section an important axiomatic system will be presented which is less general than a generalised information algebra, but which induces one under certain circumstances, and which has many models or instances.

3.2 Valuation Algebras

Consider the quasi-separoid $(D; \leq, \perp_L)$, where D is a lattice. If $(\Phi, D; \leq, \perp_L, d, \cdot, t)$ is a (generalised) information algebra, then an alternative Axiomatic system can be derived. The operations of labeling and combination remain the same, whereas the transport operation is replaced by the partial operation

$$\pi_y(\phi) = t_y(\phi), \text{ defined for } y \leq d(\phi).$$

This operation is called *projection*, sometimes also *marginalisation*. The Semigroup and Labeling Axioms remain. Axiom A3 is reformulated in the following way:

1. If $d(\phi) = x$, then $\phi \cdot 0_x = 0_x$, $\phi \cdot 1_x = \phi$,
2. If $y \leq x = d(\phi)$, then $\pi_y(\phi) = 0_y$ if and only if $\phi = 0_x$,
3. If $y \leq x$, then $\pi_y(1_x) = 1_x$,
4. $1_x \cdot 1_y = 1_{x \vee y}$.

For $x \leq y \leq z = d(\phi)$, we obtain,

$$\pi_x(\phi) = t_x(\phi) = t_x(t_y(\phi)) = \pi_x(\pi_y(\phi)).$$

This tells that projection can be executed stepwise. Further if $d(\phi) = x$, $d(\psi) = y$, then (Lemma 3.1),

$$\pi_x(\phi \cdot \psi) = t_x(\phi \cdot \psi) = \phi \cdot t_x(\psi).$$

But we have also $x \perp_L y | x \wedge y$. Then A4 and A3 imply $t_x(\psi) = t_x(t_{x \wedge y}(\psi)) = t_{x \wedge y}(\psi) \cdot 1_x$. Introducing this above, yields

$$\pi_x(\phi \cdot \psi) = \phi \cdot \pi_{x \wedge y}(\psi) \cdot 1_x = \phi \cdot \pi_{x \wedge y}(\psi).$$

In summary, we have a system $(\Phi, D; \leq, d, \cdot, \pi)$ with the three operations of labeling, combination and projection, satisfying the following conditions:

S0 Lattice: (D, \leq) is a lattice.

S1 Semigroup: (Φ, \cdot) is a commutative semigroup.

S2 Labeling: $d(\phi \cdot \psi) = d(\phi) \vee d(\psi)$ and $d(\pi_x(\phi)) = x$.

S3 Unit and Null: For all $x \in D$ there are elements 0_x and 1_x with $d(0_x) = d(1_x) = x$ and such that

1. If $d(\phi) = x$, then $\phi \cdot 0_x = 0_x$ and $\phi \cdot 1_x = \phi$,
2. If $y \leq x = d(\phi)$, then $\pi_y(\phi) = 0_y$ if and only if $\phi = 0_x$,
3. If $y \leq x$, then $\pi_y(1_x) = 1_y$,
4. $1_x \cdot 1_y = 1_{x \vee y}$.

S4 Projection: If $x \leq y \leq d(\phi)$, then

$$\pi_x(\phi) = \pi_x(\pi_y(\phi)). \quad (3.4)$$

S5 Combination: If $d(\phi) = x$ and $d(\psi) = y$, then

$$\pi_x(\phi \cdot \psi) = \phi \cdot \pi_{x \wedge y}(\psi). \quad (3.5)$$

This corresponds essentially to the Axioms introduced in (Shenoy & Shafer, 1990a) for multivariate models, except that Axiom S3 is there missing. A system $(\Phi, D; \leq, d, \cdot, \pi)$ like this has also been called a *(labeled) valuation algebra* (Kohlas & Shenoy, 2000; Kohlas, 2003a). A general information algebra with respect to the q-separoid $(D; \leq, \perp_L)$ induces therefore a valuation algebra. The converse is also the case, as we shall see below.

If the Idempotency Axiom

S6 Idempotency: If $x \leq d(\phi)$, then

$$\phi \cdot \pi_x(\phi) = \phi \quad (3.6)$$

is added, then a valuation algebra is also called a (labeled) information algebra in (Kohlas, 2003a). This is then a special case of an idempotent generalised information algebra in the present sense. There are many examples or instances of valuation algebras known. Initial examples include belief

and possibility functions, relational algebra (relational databases). Probability potentials, related to Bayesian networks, satisfy the Axioms without S3 (Shenoy & Shafer, 1990a). We refer to (Kohlas, 2003a; Pouly & Kohlas, 2011) for much more models of valuation algebras.

There are various alternative Axiomatic systems for valuation algebras. In particular, there are valuation algebras, where Axiom S3 is removed, that is no unit and null elements are assumed. Or we consider also valuation algebras where a weaker Axiom about unit elements is assumed:

S3' For all $x \in D$ there are elements 1_x with $d(1_x) = x$ and such that

1. If $d(\phi) = x$, then $\phi \cdot 1_x = \phi$.
2. $1_x \cdot 1_y = 1_{x \vee y}$.

Further, there are valuation algebras, where the Combination Axiom S5 is satisfied in a stronger version:

S5' If $d(\phi) = x$, $d(\psi) = y$, and $x \leq z \leq x \vee y$, then

$$\pi_z(\phi \cdot \psi) = \phi \cdot \pi_{y \wedge z}(\psi).$$

Item 3 of Axiom S3 is called *stability* and a valuation algebra satisfying S3 is called *stable*. It is essential to recover a (generalised) information algebra from a valuation algebra, as will be seen below. If S3 is removed, or replaced by S3', stability is lost, and there is no generalised information algebra associated with the valuation algebra. Note also that if the lattice $(D; \leq)$ is distributive, then S5' follows from S5. In fact, we have, if $d(\phi) = x$, $d(\psi) = y$, and $x \leq z \leq x \vee y$,

$$\phi \cdot \psi = \phi \cdot \psi \cdot 1_{x \vee y} = \phi \cdot \psi \cdot 1_z \cdot 1_{x \vee y} = \phi \cdot \psi \cdot 1_z.$$

Therefore, we obtain from the Combination Axiom S5,

$$\pi_z(\phi \cdot \psi) = \pi_z((\phi \cdot 1_z) \cdot \psi) = (\phi \cdot 1_z) \cdot \pi_{y \wedge z}(\psi).$$

But distributivity of the lattice $(D; \leq)$ implies $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) = z$, so that indeed $\pi_z(\phi \cdot \psi) = \phi \cdot \pi_{y \wedge z}(\psi)$.

Note finally that, if $d(\phi) = x$, $d(\psi) = y$ and $x \leq z \leq y$, then S5' implies that $\pi_z(\phi \cdot \psi) = \phi \cdot \pi_z(\psi)$. This same result holds also for any valuation algebra with unit elements, as stated in the following lemma:

Lemma 3.4 *Let $(\Phi, D; \leq, d, \cdot, \pi)$ be a valuation algebra, either satisfying Axioms S0 to S5 or S3' instead of S3, or else without unit elements, but satisfying S5' instead of S5. In all these cases, if $d(\phi) = x$, $d(\psi) = y$ and $x \leq z \leq y$, then*

$$\pi_z(\phi \cdot \psi) = \phi \cdot \pi_z(\psi).$$

Proof. In the case of existing units satisfying S3 or S3' we have $\pi_z(\phi \cdot \psi) = \pi_z((\phi \cdot 1_z) \cdot \psi)$ and then by the Combination Axiom S5, $\pi_z((\phi \cdot 1_z) \cdot \psi) = \phi \cdot 1_z \cdot \pi_z(\psi) = \phi \cdot \pi_z(\psi)$. \square

We show now that any *stable* valuation algebra induces a general information algebra. So, let $(\Phi, D; \leq, d, \cdot, \pi)$ be such a valuation algebra. We consider the q-separoid $(D; \leq, \perp_L)$ where D is a lattice, and where $x \perp_L y | z$ iff $(x \vee z) \wedge (y \vee z) = z$, see Section 2.1. We are going to extend the projection operation π , which is a partial transport operation defined only for domains $x \leq d(\phi)$ to a full transport operation in two steps. First, for $y \geq d(\phi)$ we define

$$e_y(\phi) = \phi \cdot 1_y.$$

Then $e_y(\phi)$ has label y . It is thus an extension of ϕ to a larger domain. By the Combination Axiom S5 and by Axiom S3, $\pi_x(e_y(\phi)) = \pi_x(\phi \cdot 1_y) = \phi \cdot \pi_x(1_y) = \phi \cdot 1_x = \phi$ (stability!). So we see that we may recover ϕ from its extension. The extension $e_y(\phi)$ is therefore called the *vacuous extension* (Shafer, 1991; Kohlas, 2003a). Then, the transport operation is defined for any $y \in D$ by first vacuously extending ϕ from its domain x to $x \vee y$ and then projecting this extension back to y . So, we define

$$t_y(\phi) = \pi_y(e_{x \vee y}(\phi)), \text{ where } d(\phi) = x. \quad (3.7)$$

Note that for $y \leq d(\phi)$ the transport operation is projection, $t_y(\phi) = \pi_y(\phi)$ and for $y \geq d(\phi)$ it is vacuous extension, $t_y(\phi) = e_y(\phi)$. We remark that the transport operation can equivalently also be defined by

$$t_y(\phi) = e_y(\pi_{x \wedge y}(\phi)). \quad (3.8)$$

For a proof of this alternative way to compute the transport operation, we refer to (Kohlas, 2003a). A useful property of the transport operation as defined above, is that if $y \leq z$, then

$$t_y(\phi) = t_y(t_z(\phi)). \quad (3.9)$$

Again we refer to (Kohlas, 2003a) for the proof.

We are now going to show, that this transport operation in a valuation algebra satisfies conditions A3, A4, A5 and A6 of a generalised information algebra. Axioms A0, A1, A2 are inherited directly from the valuation algebra.

Theorem 3.1 *Let $(\Phi, D; \leq, d, \cdot, \pi)$ be a stable valuation algebra. Then*

1. $t_y(\phi) = 0_y$ if and only if $\phi = 0_{d(\phi)}$,
2. $d(\phi) = y$ implies $\phi \cdot 1_x = t_{x \vee y}(\phi)$.

Proof. 1.) Assume first that $\phi = 0_x$. Then, by the Combination Axiom S5 and Axiom S3, $t_x(t_{x \vee y}(0_x)) = \pi_x(t_{x \vee y}(0_x)) = \pi_x(0_x \cdot 1_{x \vee y}) = 0_x \cdot 1_x = 0_x$, hence by Axiom S3, $t_{x \vee y}(0_x) = 0_{x \vee y}$. But, again by Axiom S3, $t_y(0_{x \vee y}) = \pi_y(0_{x \vee y}) = 0_y$ and using Axiom S5, it follows that $\pi_y(0_{x \vee y}) = \pi_y(0_y \cdot 1_{x \vee y}) = 0_y \cdot \pi_y(1_{x \vee y}) = 0_y \cdot 1_y = 0_y$.

Conversely, assume $d(\phi) = x$ and $t_y(\phi) = 0_y$. Then $t_y(\phi) = \pi_y(t_{x \vee y}(\phi)) = 0_y$. From Axiom S3 we obtain then $t_{x \vee y}(\phi) = 0_{x \vee y}$. Again by S3, $\phi = \pi_x(t_{x \vee y}(\phi)) = \pi_x(0_{x \vee y}) = 0_x$.

2.) We have by Axioms S2 and S3 $\phi \cdot 1_x = \phi \cdot 1_x \cdot 1_{x \vee y} = \phi \cdot 1_{x \vee y} = t_{x \vee y}(\phi)$. \square

This shows that Axiom A3 is valid. Axioms A4, A5 and A6 are asserted in the following theorem

Theorem 3.2 *Let $(\Phi, D; \leq, d, \cdot, \pi)$ be a stable valuation algebra. Then*

1. $x \perp_L y | z$ and $d(\phi) = x$ imply $t_y(\phi) = t_y(t_z(\phi))$.
2. $x \perp_L y | z$ and $d(\phi) = x$, $d(\psi) = y$ imply $t_z(\phi \cdot \psi) = t_z(\phi) \cdot t_z(\psi)$.
3. $d(\phi) = x$ implies $t_x(\phi) = \phi$.

Proof. 1.) Assume $d(\phi) = x$. Define $\psi = t_{x \vee z}(\phi)$, so that $d(\psi) = x \vee z$. By the alternative definition of the transport operation (3.8), and $x \perp_L y | z$,

$$t_{y \vee z}(\psi) = e_{y \vee z}(\pi_{(x \vee z) \wedge (y \vee z)}(\psi)) = e_{y \vee z}(\pi_z(\psi)) = t_{y \vee z}(t_z(\psi)). \quad (3.10)$$

Now, by the definition of the transport operation and reversibility of vacuous extension, we have

$$t_y(\phi) = \pi_y(e_{x \vee y}(\phi)) = \pi_y(\pi_{x \vee y}(e_{x \vee y \vee z}(e_{x \vee y}(\phi)))).$$

As projection, vacuous extension can also be executed stepwise, see Axiom S4 and (3.9), such that

$$t_y(\phi) = \pi_y(e_{x \vee y \vee z}(\phi)) = \pi_y(\pi_{y \vee z}(e_{x \vee y \vee z}(e_{x \vee z}(\phi)))) = \pi_y(t_{y \vee z}(t_{x \vee z}(\phi))).$$

Using (3.10), we obtain

$$t_y(\phi) = t_y(t_{y \vee z}(t_z(t_{x \vee z}(\phi)))).$$

Since $z \leq y \vee z, x \vee z$, we use (3.9) twice to conclude finally

$$t_y(\phi) = t_y(t_z(t_{x \vee z}(\phi))) = t_y(t_z(\phi)).$$

This proves the first part of the theorem.

2.) Assume $d(\phi) = x$ and $d(\psi) = y$. Using Axiom S3, it follows that $\phi \cdot \psi \cdot 1_z = \phi \cdot \psi \cdot 1_{x \vee y} \cdot 1_z = \phi \cdot \psi \cdot 1_{x \vee y \vee z} = e_{x \vee y \vee z}(\phi \cdot \psi)$. So, by definition of the transport operation,

$$t_z(\phi \cdot \psi) = \pi_z(\phi \cdot \psi \cdot 1_z) = \pi_z((\phi \cdot 1_z) \cdot (\psi \cdot 1_z)).$$

Now by Axiom S4, since $z \leq x \vee z \leq x \vee y \vee z$,

$$t_z(\phi \cdot \psi) = \pi_z(\pi_{x \vee z}((\phi \cdot 1_z) \cdot (\psi \cdot 1_z))).$$

As $d(\phi \cdot 1_z) = x \vee z$ and $d(\psi \cdot 1_z) = y \vee z$, we may apply S5 and obtain, using $x \perp_L y|z$,

$$t_z(\phi \cdot \psi) = \pi_z((\phi \cdot 1_z) \cdot \pi_z(\psi \cdot 1_z)).$$

Again, by S5, and then the definition of the transport operation we conclude

$$t_z(\phi \cdot \psi) = \pi_z(\phi \cdot 1_z) \cdot \pi_z(\psi \cdot 1_z) = t_z(\phi) \cdot t_z(\psi).$$

This proves the second item of the theorem.

3.) If $d(\phi) = x$, then $d(t_x(\phi)) = d(\pi_x(e_x(\phi))) = d(e_x(\phi)) = x$, by Axiom S2 and the definition of vacuous extension. This verifies the third item of the theorem. \square

In summary, we have shown that a stable valuation algebra induces a generalised information algebra.

Theorem 3.3 *If $(P, D; \leq, d, \cdot, \pi)$ is a stable valuation algebra, then $(\Phi, D; \leq, \perp_L, d, \cdot, t)$, where the operation t is defined by (3.7) is a generalised information algebra.*

We give here and in the next section important examples of valuation algebras, of different sorts. More example may be found in (Pouly & Kohlas, 2011).

Example 3.3 Densities: This example is based on the multivariate model of domains (see Section 2.2). Let $(D; \subseteq)$ be the lattice of finite subsets of $\omega = \{1, 2, \dots\}$. We consider here the linear vector spaces \mathbb{R}^s of real valued tuples $x : s \rightarrow \mathbb{R}$ where s is a finite subset of ω . On a space \mathbb{R}^s we consider nonnegative functions $f : \mathbb{R}^s \rightarrow \mathbb{R}^+ \cup \{0\}$, whose integrals

$$\int_{-\infty}^{+\infty} f(x) dx \quad (3.11)$$

exist and are finite. To simplify, we consider continuous functions and Riemann integrals; it would also be possible to consider measurable functions and Lebesgue integrals (Kohlas, 2003a). Such functions are called *densities* on \mathbb{R}^s . We define the operations of a valuation algebra for densities as follows:

1. *Labeling:* $d(f) = s$ if f is a density on \mathbb{R}^s .
2. *Combination:* For densities f and g with $d(f) = s$ and $d(g) = t$ and $x \in \mathbb{R}^{s \cup t}$,

$$(f \cdot g)(x) = f(x_s) \times g(x_t),$$

where x_s and x_t denote the tuple of components of x in s and t respectively.

3. *Projection:* For density f with $d(f) = s$, $t \subseteq s$, and $x \in \mathbb{R}^t$,

$$(\pi_t(f))(x_t) = \int_{-\infty}^{+\infty} f(x) dx_{s-t}.$$

It is straightforward to verify the Axioms of a valuation algebra for this system, except Axioms S3 or S3'. There are no unit elements, since the function $h(x) = 1$ for all $x \in \mathbb{R}^s$ is not finitely integrable, hence no density. However, the strong Combination Axiom S5' is satisfied for densities. This valuation algebra is related to probabilistic nets of conditional density functions (see (Kohlas, 2003a)). \ominus

Here follows another example of a valuation algebra.

Example 3.4 *Gaussian Potentials:* Gaussian densities are of particular interest in applications. A multivariate Gaussian density over a set s of variables is defined by

$$f(x) = (2\pi)^{-n/2} (\det \Sigma)^{-1/2} e^{-(1/2)(x-\mu)' \Sigma^{-1} (x-\mu)}.$$

Here μ is a vector in \mathbb{R}^s (see previous example), and Σ is a symmetric positive definite $s \times s$ matrix. The vector μ is the expected value vector of the density and Σ the variance-covariance matrix. The matrix $\mathbf{K} = \Sigma^{-1}$ is called the concentration matrix of the density. It is also symmetric and positive definite. A Gaussian density may be represented or determined by the pair (μ, \mathbf{K}) . Each such pair with μ an s -vector and \mathbf{K} a symmetric positive definite $s \times s$ matrix determines a Gaussian density. Gaussian densities belong to the valuation algebra of densities defined in the previous example. Labeling, combination and projection can however now be expressed in terms of the pairs (μ, \mathbf{K}) . For this purpose, if $t \supseteq s$ and μ is an s -vector, \mathbf{K} a $s \times s$ matrix let $\mu^{\uparrow t}$ and $\mathbf{K}^{\uparrow t}$ be the vector or matrix obtained by adding to μ and \mathbf{K} 0-entries for all indices in $t - s$. Further, if $t \subseteq s$, then let μ_t and $\mathbf{K}_{t,t}$ be the subvector or submatrix of μ and \mathbf{K} respectively with components in t .

We then define the following operations on pairs (μ, \mathbf{K}) :

1. *Labeling:* $d(\mu, \mathbf{K}) = s$ if μ is a s -vector and \mathbf{K} a $s \times s$ matrix.
2. *Combination:* For pairs (μ_1, \mathbf{K}_1) and (μ_2, \mathbf{K}_2) with $d(\mu_1, \mathbf{K}_1) = s$ and $d(\mu_2, \mathbf{K}_2) = t$,

$$(\mu_1, \mathbf{K}_1) \cdot (\mu_2, \mathbf{K}_2) = (\mu, \mathbf{K})$$

with

$$\mathbf{K} = \mathbf{K}_1^{\uparrow s \cup t} + \mathbf{K}_2^{\uparrow s \cup t}$$

and

$$\mu = \mathbf{K}^{-1} \left(\mathbf{K}_1^{\uparrow s \cup t} \mu_1^{\uparrow s \cup t} + \mathbf{K}_2^{\uparrow s \cup t} \mu_2^{\uparrow s \cup t} \right).$$

3. *Projection:* For a pair (μ, \mathbf{K}) with $d(\mu, \mathbf{K}) = s$, $t \subseteq s$,

$$\pi_t(\mu, \mathbf{K}) = (\mu_t, ((\mathbf{K}^{-1})_{t,t})^{-1}).$$

This is justified by the fact, that the combination of two Gaussian densities results again in a Gaussian density (after normalisation), and so does projection of Gaussian density. We refer to (Kohlas, 2003a) for more details. Again, the algebra of Gaussian potentials has no unit elements, but satisfies the strong Combination Axiom S5'. As for densities, this valuation algebra is useful for local computation with nets of conditional Gaussian density functions. For another application of this valuation algebra to linear systems with Gaussian disturbances we refer to (Pouly & Kohlas, 2011). \ominus

Most of the studies about local computation have been made with respect to valuation algebras in the multivariate framework, as discussed for instance in (Lauritzen & Spiegelhalter, 1988; Shenoy & Shafer, 1990a; Shenoy & Shafer, 1990a; Kohlas, 2003a). We shall show in Section 4 how local computation can be defined in the present more general framework of generalised information algebras. More examples, in fact a whole generic class of examples of valuation and information algebras, and not only in the multivariate framework, is given in the next section.

3.3 Semiring Information Algebras

In this section we introduce a large class of information and valuation algebras based on semirings. This is an extension of the class of algebras described in (Kohlas & Wilson, 2006) based on of multivariate models. The elements of the algebra, the pieces of information, will be mappings from a frame of a family of compatible frames to a semiring. Many well-known instances of valuation or information algebras are of this kind.

To start we recall the definition of semirings.

Definition 3.1 *Semirings;* Let A be any set with two binary operations $+$: $A \times A \rightarrow A$ and \times : $A \times A \rightarrow A$. Then the signature $(A; +, \times)$ is a commutative semiring, if

1. $(A; +)$ and (A, \times) are commutative semigroups,
2. \times distributes over $+$, that is $a \times (b + c) = (a \times b) + (a \times c)$ for all $a, b, c \in A$.

In general for a semiring $(A; +, \times)$ multiplication is not necessarily assumed to be commutative; but we shall here only consider commutative semirings. So, in the sequel when we speak of semirings, we always assume them to be commutative. The concept of a semiring is a weakening of the

concept of a ring. Rings and fields are all also semirings. Further examples will be mentioned below.

If there exists an element $0 \in A$ such that $a + 0 = a$ and $a \times 0 = 0$, then 0 is called a null element and the semiring $(A; +, \times, 0)$ a semiring with null element. A null element is always unique and if A has no null element it can always be adjoined. If there exists an element $1 \in A$ such that $1 \times a = a \times 1 = a$, then 1 is called a unit element and the semiring $(A; +, \times, 1)$ a semiring with unit element. If $(A; +, \times, 0)$ is a semiring with null element and $a + b = 0$ implies always $a = b = 0$, then A is called *positive*. If addition is *idempotent* in a semiring $(A; +, \times)$, that is $a + a = a$ for all $a \in A$, then a unit element can be adjoined (see (Kohlas & Wilson, 2006)). If $(A; +, \times, 1)$ has a unit element such that $1 + 1 = 1$, then $a + a = a \times (1 + 1) = a$ and addition is idempotent. For instance, if $(A, +, \times, 0, 1)$ is a semiring with null and unit elements such that $a + 1 = 1$, then addition is idempotent. In this case A is called a *c-semiring* (constraint semiring). Finally, if A is a c-semiring and multiplication is also idempotent, then $(A, +, \times, 0, 1)$ is a bounded distributive lattice with addition as join and multiplication as meet.

Here follow a few examples of important semirings.

Example 3.5 *Arithmetic Semirings:*

Integers, rational numbers and real numbers form semirings (the latter two even fields) under ordinary addition and multiplication. Nonnegative or positive integers, rational or real number are still semirings, these semirings are positive and they have all have unit elements and null elements (to be adjoined for positive numbers). \ominus

Example 3.6 *Boolean Semiring:* Consider $A = \{0, 1\}$ and define the operation $+$ as $a + b = \max\{a, b\}$ and \times as $a \times b = \min\{a, b\}$. This is a semiring with 0 as null and 1 as unit element. In addition $0 + 1 = 1 + 1 = 1$, so it is a c-semiring. It is used to describe the valuation algebra of constraint systems and relational algebra (Kohlas, 2003a; Kohlas & Wilson, 2006). \ominus

Further examples, including Bottleneck algebras, $(\max / \min, +)$ -semirings, t-norms, distributive lattices are to be found in (Kohlas & Wilson, 2006; Pouly & Kohlas, 2011) and of course any book on semirings.

Now, we introduce semiring valuations. As a preparation consider finite, disjoint index sets I_j for $j = 1, \dots, n$ and let $I = I_1 \cup \dots \cup I_n$. then, due to commutativity and associativity of addition,

$$\sum_{j=1}^n \sum_{i \in I_j} a_i = \sum_{i \in I} a_i.$$

This will be used in the sequel without further reference.

Let now $(\mathcal{F}, \mathcal{R})$ be a family of compatible frames, whose frames $\Theta \in \mathcal{F}$ are all finite. Let further the relation $\Theta_1 \perp \Theta_2 | \Lambda$ be the associated conditional independence relation forming a q-separoid. We call the elements θ of a frame Θ its atoms. If Λ is a coarsening of Θ and $\tau : \Lambda \rightarrow \Theta$ the corresponding refining, then there is exactly one atom λ in Λ , compatible with θ , namely the atom λ such that $\theta \in \tau(\lambda)$. In other words, we have $v(\{\theta\}) = \{\lambda\}$. We define now

$$t_\Lambda(\theta) = v(\{\theta\}) \quad (3.12)$$

It is called the *projection* of the atom θ to the coarser frame Λ . Suppose next, that Λ is a refinement of Θ with the refining $\tau : \Theta \rightarrow \Lambda$. Now, the atoms in $\tau(\theta)$ are all compatible with θ and we define

$$t_\Lambda(\theta) = \tau(\theta). \quad (3.13)$$

In the general case of two arbitrary frames Θ and Λ , the atoms λ in Λ are compatible with an atom θ in the frame Θ , if $\lambda \in R_\theta(\Lambda) = \{\lambda : \tau(\theta) \cap \mu(\lambda) \neq \emptyset\}$, where τ and μ are the refinings of Θ and Λ to $\Theta \vee \Lambda$ respectively (see Section 2.3). Therefore, we define in the general case

$$t_\Lambda(\theta) = v(\tau(\theta)) = R_\theta(\Lambda), \quad (3.14)$$

where v is the saturation operator associated with the refining μ . This is the restriction of the general transport operation of subsets of frames to one-element sets (see Section 3.1). Since the subset algebra of a f.c.f is a generalised information algebra, it satisfies the transport axiom A4 (Section 3.1). We restate this axiom as an important result for the transport of atoms:

Theorem 3.4 *Assume $\theta \in \Theta$ and $\Theta \perp \Lambda_1 | \Lambda_2$. then*

$$t_{\Lambda_1}(\theta) = t_{\Lambda_1}(t_{\Lambda_2}(\theta)). \quad (3.15)$$

These transport operations become important below. Since a frame can be considered as representing possible answers to some question, atoms being precise answers, these transport operations of atoms determine answers in frames Λ , compatible with a precise answer θ in a frame Θ . So transport of atoms has an important information-theoretic meaning.

For a semiring $(A; +, \times)$ we define now A -valuations on a frame Θ to be mappings

$$\phi : \Theta \rightarrow A$$

from a frame Θ of \mathcal{F} into the semiring A . Let Φ_Θ be the set of all A -valuations on the frame Θ and define

$$\Phi = \bigcup_{\Theta \in \mathcal{F}} \Phi_\Theta,$$

the set of all A-valuations in the f.c.f. $(\mathcal{F}, \mathcal{R})$. Such A-valuations have been considered in (Kohlas & Wilson, 2006) for the special case of multivariate models. The results obtained there can be extended to the present more general case of A-valuations in a f.c.f. First, we define the following operations for A-valuations, using the transport operations of atoms defined above:

1. *Labeling*: $d(\phi) = \Theta$ if ϕ is an A-valuation on Θ .
2. *Combination*: If ϕ, ψ are A-valuations with $d(\phi) = \Theta$ and $d(\psi) = \Lambda$, then for $\zeta \in \Theta \vee \Lambda$, an A-valuation $\phi \cdot \psi$ on frame $\Theta \vee \Lambda$ is defined by

$$\phi \cdot \psi(\zeta) = \phi(t_\Theta(\zeta)) \times \psi(t_\Lambda(\zeta)). \quad (3.16)$$

3. *Transport*: If ϕ is a A-valuation with $d(\phi) = \Theta$ and $\lambda \in \Lambda$, then an A-valuation $t_\Lambda(\phi)$ on frame Λ is defined by

$$t_\Lambda(\phi)(\lambda) = \sum_{\theta \in t_\Theta(\lambda)} \phi(\theta). \quad (3.17)$$

The idea behind combination is that in the combined A-valuation $\phi \cdot \psi$ the value of ζ in the supremum or the combined frame $\Theta \vee \Lambda$ equals the product of the values of the compatible atoms in the frames Θ and Λ respectively. For transport of an A-valuation to a frame Λ , the value of an atom in Λ is the sum of the values of the compatible atoms in the original frame Θ . We shall see in examples below that this makes sense in all applications.

How far are the properties of an information algebra satisfied by these operations of A-valuations? Clearly, the Labeling Axiom A2 of an information algebra is satisfied by the definition of combination and transport,

$$d(\phi \cdot \psi) = d(\phi) \vee d(\psi), \quad d(t_\Lambda(\phi)) = \Lambda.$$

Further, by assumption $(\mathcal{F}; \leq, \perp)$ is a q-separoid, hence Axiom A0 is valid. The Identity Axiom A6 is also valid since if $d(\phi) = \Theta$, then $t_\Theta(\theta) = \theta$ for all $\theta \in \Theta$. Next, we show that combination is both commutative and associative.

Theorem 3.5 *With the definition of combination above, $(\Phi; \cdot)$ is a commutative semigroup.*

Proof. Commutativity of combination follows by the definition of combination from commutativity of semigroup multiplication.

To show associativity consider A-valuations ϕ_1, ϕ_2 and ϕ_3 on frames Θ_1, Θ_2 and Θ_3 respectively. Let θ be an atom in $\Theta_1 \vee \Theta_2 \vee \Theta_3$. Then we have

$$\begin{aligned} & (\phi_1 \cdot \phi_2) \cdot \phi_3(\theta) \\ &= \phi_1 \cdot \phi_2(t_{\Theta_1 \vee \Theta_2}(\theta)) \times \phi_3(t_{\Theta_3}(\theta)) \\ &= (\phi_1(t_{\Theta_1}(t_{\Theta_1 \vee \Theta_2}(\theta)))) \times \phi_2(t_{\Theta_2}(t_{\Theta_1 \vee \Theta_2}(\theta)))) \times \phi_3(t_{\Theta_3}(\theta)) \end{aligned}$$

But from $\Theta_1 \perp \Theta_2 | \Theta_1 \vee \Theta_2$ it follows (Theorem 3.4) that

$$t_{\Theta_1}(t_{\Theta_1 \vee \Theta_2}(\theta)) = t_{\Theta_1}(\theta), \quad t_{\Theta_2}(t_{\Theta_1 \vee \Theta_2}(\theta)) = t_{\Theta_2}(\theta).$$

So, we obtain

$$(\phi_1 \cdot \phi_2) \cdot \phi_3(\theta) = (\phi_1(t_{\Theta_1}(\theta)) \times \phi_2(t_{\Theta_2}(\theta))) \times \phi_3(t_{\Theta_3}(\theta)).$$

We obtain in the same way

$$\phi_1 \cdot (\phi_2 \cdot \phi_3)(\theta) = \phi_1(t_{\Theta_1}(\theta)) \times (\phi_2(t_{\Theta_2}(\theta)) \times \phi_3(t_{\Theta_3}(\theta))). \quad (3.18)$$

Since multiplication in A is associative, it follows that $(\phi_1 \cdot \phi_2) \cdot \phi_3 = \phi_1 \cdot (\phi_2 \cdot \phi_3)$. \square

So, the Semigroup Axiom A2 is satisfied. It is evident that each $(\Phi_\Theta; \cdot)$ is a subsemigroup of $(\Phi; \cdot)$.

If the semiring A has a null element 0, then the A-valuations 0_Θ defined by

$$0_\Theta(\theta) = 0 \text{ for all } \theta \in \Theta$$

are the null elements in the semigroups $(\Phi_\Theta; \cdot)$. Similarly, if the semiring A has a unit element, then the A-valuations 1_Θ defined as

$$1_\Theta(\theta) = 1 \text{ for all } \theta \in \Theta$$

are unit elements in the semigroups $(\Phi_\Theta; \cdot)$. These particular A-valuations have the following properties:

Theorem 3.6 *If the semiring A has a null element and a unit element, then*

1. $\phi \cdot 1_\Lambda = t_{\Theta \vee \Lambda}(\phi)$ if $d(\phi) = \Theta$.
2. $1_\Theta \cdot 1_\Lambda = 1_{\Theta \vee \Lambda}$.
3. $\phi \cdot 0_\Lambda = 0_{\Theta \vee \Lambda}$ if $d(\phi) = \Theta$.

Proof. 1.) Let ζ be an atom in $\Theta \vee \Lambda$. Then we have

$$\phi \cdot 1_\Lambda(\zeta) = \phi(t_\Theta(\zeta)) \times 1_\Lambda(t_\Lambda(\zeta)) = \phi(t_\Theta(\zeta)) \times 1 = \phi(t_\Theta(\zeta)) = t_{\Theta \vee \Lambda}(\zeta).$$

2.) and 3.) Again, for $\zeta \in \Theta \vee \Lambda$, we have

$$1_\Theta \cdot 1_\Lambda(\zeta) = 1_\Theta(t_\Theta(\zeta)) \times 1_\Lambda(t_\Lambda(\zeta)) = 1 \times 1 = 1.$$

and

$$\phi \cdot 0_\Lambda(\zeta) = \phi(t_\Theta(\zeta)) \times 0_\Lambda(t_\Lambda(\zeta)) = \phi(t_\Theta(\zeta)) \times 0 = 0.$$

This concludes the proof. \square

The following result is more profound and states that the Combination Axiom A5 is valid.

Theorem 3.7 Consider A -valuations ϕ and ψ with $d(\phi) = \Theta_1$ and $d(\psi) = \Theta_2$. Assume $\Theta_1 \perp \Theta_2 | \Lambda$. Then,

$$t_\Lambda(\phi \cdot \psi) = t_\Lambda(\phi) \cdot t_\Lambda(\psi).$$

Proof. Consider an atom $\lambda \in \Lambda$. Then we have by definition of transport and combination

$$\begin{aligned} t_\Lambda(\phi \cdot \psi)(\lambda) &= \sum_{\zeta \in t_{\Theta_1 \vee \Theta_2}(\lambda)} \phi \cdot \psi(\zeta) = \sum_{\zeta \in t_{\Theta_1 \vee \Theta_2}(\lambda)} \phi(t_{\Theta_1}(\zeta)) \times \psi(t_{\Theta_2}(\zeta)), \\ &= t_\Lambda(\phi) \cdot t_\Lambda(\psi)(\lambda) \end{aligned}$$

Now, $t_{\Theta_1}(\lambda) = R_\lambda(\Theta_1)$ and $t_{\Theta_2}(\lambda) = R_\lambda(\Theta_2)$. Further, we claim that

$$\begin{aligned} R_\lambda(\Theta_1, \Theta_2) &= \{(\theta_1, \theta_2) \in \Theta_1 \times \Theta_2 : \theta_1 = t_{\Theta_1}(\zeta), \theta_2 = t_{\Theta_2}(\zeta) \text{ for some } \zeta \in t_{\Theta_1 \vee \Theta_2}(\lambda)\} \end{aligned} \quad (3.19)$$

But $\Theta_1 \perp \Theta_2 | \Lambda$ implies $R_\lambda(\Theta_1, \Theta_2) = R_\lambda(\Theta_1) \times R_\lambda(\Theta_2)$. So, once the claim above is proved, we conclude that

$$\begin{aligned} t_\Lambda(\phi \cdot \psi)(\lambda) &= \sum_{\theta_1 \in R_\lambda(\Theta_1), \theta_2 \in R_\lambda(\Theta_2)} \phi(\theta_1) \times \psi(\theta_2) \\ &= \left(\sum_{\theta_1 \in R_\lambda(\Theta_1)} \phi(\theta_1) \right) \times \left(\sum_{\theta_2 \in R_\lambda(\Theta_2)} \psi(\theta_2) \right) \\ &= t_\Lambda(\phi)(\lambda) \times t_\Lambda(\psi)(\lambda) \\ &= t_\Lambda(\phi) \cdot t_\Lambda(\psi)(\lambda). \end{aligned}$$

This shows that $t_\Lambda(\phi \cdot \psi) = t_\Lambda(\phi) \cdot t_\Lambda(\psi)$.

It remains to verify the claim above. For this purpose let τ'_1 and τ'_2 be the refinings of Θ_1 and Θ_2 to $\Theta_1 \vee \Theta_2$ respectively, τ_1 and τ_2 be the refinings of Θ_1 and Θ_2 to $\Theta_1 \vee \Theta_2 \vee \Lambda$, τ the refining of Λ to $\Theta_1 \vee \Theta_2 \vee \Lambda$ and μ the refining of $\Theta_1 \vee \Theta_2$ to $\Theta_1 \vee \Theta_2 \vee \Lambda$ and finally v the outer reduction associated with μ . Then by definition

$$R_\lambda(\Theta_1, \Theta_2) = \{(\theta_1, \theta_2) : \tau_1(\theta_1) \cap \tau_2(\theta_2) \cap \tau(\lambda) \neq \emptyset\}.$$

Now, $\tau_1(\theta_1) \cap \tau_2(\theta_2) \cap \tau(\lambda) \neq \emptyset$ is equivalent to

$$\mu(\tau'_1(\theta_1)) \cap \mu(\tau'_2(\theta_2)) \cap \tau(\lambda) = \mu(\tau'_1(\theta_1) \cap \tau'_2(\theta_2)) \cap \tau(\lambda) \neq \emptyset.$$

Consider an atom $\zeta \in t_{\Theta_1 \vee \Theta_2}(\lambda)$ and atoms $\theta_1 = t_{\Theta_1}(\zeta)$, $\theta_2 = t_{\Theta_2}(\zeta)$ such that (θ_1, θ_2) belongs to the set on the right hand side of (3.19). This means

that $\zeta \in \tau'_1(\theta_1) \cap \tau'_2(\theta_2) = \tau'_1(t_{\Theta_1}(\zeta)) \cap \tau'_2(t_{\Theta_2}(\zeta))$ and $\mu(\zeta) \cap \tau(\lambda) \neq \emptyset$. From this it follows that

$$\mu(\tau'_1(t_{\Theta_1}(\zeta)) \cap \tau'_2(t_{\Theta_2}(\zeta))) \cap \tau(\lambda) \supseteq \mu(\zeta) \cap \tau(\lambda) \neq \emptyset.$$

So, $(\theta_1, \theta_2) \in R_\lambda(\Theta_1, \Theta_2)$ and the right hand side of (3.19) is contained in $R_\lambda(\Theta_1, \Theta_2)$.

Conversely, consider a pair $(\theta_1, \theta_2) \in R_\lambda(\Theta_1, \Theta_2)$ so that $\mu(\tau'_1(\theta_1) \cap \tau'_2(\theta_2)) \cap \tau(\lambda) \neq \emptyset$. This implies $\tau'_1(\theta_1) \cap \tau'_2(\theta_2) \neq \emptyset$. So, we may select an atom $\omega \in \mu(\tau'_1(\theta_1) \cap \tau'_2(\theta_2)) \cap \tau(\lambda)$ and then an atom $\zeta \in v(\omega) \subseteq \tau'_1(\theta_1) \cap \tau'_2(\theta_2)$. But this means that $\theta_1 = t_{\Theta_1}(\zeta)$ and $\theta_2 = t_{\Theta_2}(\zeta)$ and $\omega \in \mu(\zeta) \cap \tau(\lambda) \neq \emptyset$, which implies $\zeta \in t_{\Theta_1 \vee \Theta_2}(\lambda)$. So $R_\lambda(\Theta_1, \Theta_2)$ is contained in the right hand side of (3.19) and this proves the claim (3.19). \square

So far, we have verified that A-valuations satisfy some axioms of a generalised information algebra. However, the Transport Axiom A4 needs an additional property of the semiring. And further, the property that $t_\Lambda(1_\Theta) = 1_\Lambda$ (see Lemma 3.1) is not necessarily satisfied for an arbitrary semiring. A sufficient condition for this is that addition in a semiring $(A; +, \times, 0, 1)$ is *idempotent*. Then we have from the definition of transport

$$t_\Lambda(1_\Theta)(\lambda) = \sum_{\theta \in t_\Theta(\lambda)} 1 = 1.$$

And this condition is also sufficient for the validity of the Transport Axiom A4.

Theorem 3.8 *If addition in the semiring $(A; +, \times, 0, 1)$ is idempotent, then for all A-valuations ϕ with $d(\phi) = \Theta$, $\Theta \perp \Lambda_1 | \Lambda_2$ implies*

$$t_{\Lambda_1}(\phi) = t_{\Lambda_1}(t_{\Lambda_2}(\phi)).$$

Proof. For an atom $\lambda \in \Lambda_1$ we have

$$t_{\Lambda_1}(t_{\Lambda_2}(\phi))(\lambda) = \sum_{\lambda' \in t_{\Lambda_2}(\lambda)} t_{\Lambda_2}(\phi)(\lambda') = \sum_{\lambda' \in t_{\Lambda_2}(\lambda)} \sum_{\theta \in t_\Theta(\lambda')} \phi(\theta),$$

Now, $\Theta \perp \Lambda_1 | \Lambda_2$ implies $t_\Theta(\lambda) = t_\Theta(t_{\Lambda_2}(\lambda))$ (Theorem 3.4). Let $I_{\lambda'} = t_\Theta(\lambda')$ and

$$I = \bigcup_{\lambda' \in t_{\Lambda_2}(\lambda)} I_{\lambda'}.$$

We claim that $I = t_\Theta(\lambda)$. Assume first $\theta \in I$, so that $\theta \in I_{\lambda'}$ for some $\lambda' \in t_{\Lambda_2}(\lambda)$. But then, since $t_\Theta(\lambda') \subseteq t_\Theta(\lambda)$ it follows that $\theta \in t_\Theta(\lambda)$. Conversely, consider an atom $\theta \in t_\Theta(\lambda) = t_\Theta(t_{\Lambda_2}(\lambda))$. This implies that

there is a $\lambda' \in t_{\Lambda_2}(\lambda)$ such that $\theta \in t_{\Theta}(\lambda')$, hence $\theta \in I$. So, indeed $I = t_{\Theta}(\lambda)$. It follows therefore, using idempotency of addition in the semiring A , that

$$\sum_{\lambda' \in t_{\Lambda_2}(\lambda)} \sum_{\theta \in t_{\Theta}(\lambda')} \phi(\theta) = \sum_{\theta \in t_{\Theta}(\lambda)} \phi(\theta) = t_{\Lambda_1}(\phi)(\lambda).$$

This shows that $t_{\Lambda_1}(\phi) = t_{\Lambda_1}(t_{\Lambda_2}(\phi))$. \square

In particular, if $\Theta \leq \lambda$, then $\Theta \perp \Theta | \lambda$ and therefore by this theorem $t_{\Theta}(t_{\Lambda}(\phi)) = t_{\Theta}(\phi) = \phi$ if $d(\phi) = \Theta$. This means that $t_{\Lambda}(\phi)$ may be considered as a vacuous extension of ϕ , since ϕ can be retrieved from its extension (see Section 3.2).

Now, we have nearly all Axioms A1 to A6 of a generalised information algebra. The last item missing is Axiom A 3 2.) which states that $t_{\Lambda}(\phi) = 0_{\Lambda}$ implies $\phi = 0_{\Theta}$, if $d(\phi) = \Theta$. For this it is sufficient that the semiring A is *positive*. This together with the results above, proves the following theorem:

Theorem 3.9 *Let $(A; +, \times, 0, 1)$ be a positive semiring with idempotent addition. Then $(\Phi, \mathcal{F}; \leq, \perp, d, \cdot, t)$ where Φ is the set of A -valuations, with labeling, combination and transport as defined above, is a generalised information algebra.*

We note that, even if the semiring A is not positive, then all axioms are satisfied, as long as addition is idempotent, except Axiom A 3 2.). If we scan the examples of semirings mentioned above, then we find the following examples, which induce generalised information algebras of A -valuations.

Example 3.7 Set Algebra: Consider the Boolean semiring $A = \{0, 1\}$. It is positive and addition (max) is idempotent. Here A -valuations ϕ are 0-1-functions (indicator functions) on the associated frame $d(\phi) = \Theta$. Such an indicator function defines a subset $\{\theta : \phi(\theta) = 1\}$ of the frame. Combination and transport of these indicator functions correspond exactly to combination and transport of the associated subsets as defined in Section 3.1. So, we retrieve with this Boolean semiring valuation algebra the subset algebra on a f.c.f. This covers, in the multivariate case, constraint systems and gives a subset of relational algebra (Kohlas, 2003a), which is useful in query processing and for constraint solving. Constraints may also be formulated by formulae of propositional or predicate logic. In this sense it is also related to inference in Boolean logic, see (Kohlas, 2003a). \ominus

Example 3.8 Distributive Lattices: A bounded distributive lattice A is a semiring with idempotent addition (join) and is positive. Since multiplication (meet) is also idempotent the A -valuations form in this case a proper information algebra. Such algebras are discussed in Chapter 7. Of course, the previous example is an instance of such a valuation algebra, since a Boolean algebra is a distributive lattice. We may generalise the previous

example and consider A-valuations related to any Boolean algebra A . This is related to assumption-based reasoning (Kohlas & Wilson, 2006). \ominus

Example 3.9 *Fuzzy Sets, Possibilistic Constraints:* If we take a t-norm (Kohlas, 2003a; Pouly & Kohlas, 2011) for multiplication and max for addition, then addition is idempotent and the semiring is positive. In this case an A-valuation on Θ is also called a possibilistic distribution, a possibilistic constraint or a fuzzy set. Intersection of possibilistic constraints or fuzzy sets are defined by the t-norm and addition is used to compute transport. These A-valuations form a generalised information algebra. \ominus

Example 3.10 *Optimisation:* Consider the $(\max/\min, +)$ semiring of reals. Again max and min are idempotent. So, the A-valuations form a generalised information algebra although A is not positive, satisfying all axioms except A 3 2.). We may always adjoin a least element \perp to a semilattice $(\mathcal{F}; \leq)$ and define

$$t_{\perp}(\phi) = \sum_{\theta \in \Theta} \phi(\theta).$$

if ϕ is an A-valuation on Θ . In the present particular case, we have then

$$t_{\perp}(\phi_1 \cdot \dots \cdot \phi_n)(\theta) = \max_{\theta \in \Theta} (\phi_1(t_{\Theta}(\theta)) + \dots + \phi_n(t_{\Theta}(\theta))).$$

So this information algebra and its associated local computation scheme serves for optimisation. Note that local computation yields the maximum value of the combination. But the scheme may be adopted to compute also minimal solutions (Shenoy, 1996; Pouly & Kohlas, 2011). Local computation is then a version of dynamic programming. \ominus

Now we change the focus on A-valuations to some extent with the goal to obtain *valuation algebras* instead of generalised information algebras. We consider f.c.f $(\mathcal{F}, \mathcal{R})$ for which we assume that

1. $(\mathcal{F}; \leq)$ is a lattice,
2. the relation $\Theta \perp \Lambda | \Theta \wedge \Lambda$ holds for all pairs of frames $\Theta, \Lambda \in \mathcal{F}$.

It is interesting and important to clarify what these restricting requirements mean. Let μ_1 and μ_2 be the refinings of $\Theta \wedge \Lambda$ to Θ and Λ respectively and τ_1 and τ_2 the refinings of Θ and Λ to $\Theta \vee \Lambda$. Consider atoms θ, λ and ζ in Θ, Λ and $\Theta \wedge \Lambda$ respectively. Note that $\tau_1(\theta) \cap \tau_1(\mu_1(\zeta)) \neq \emptyset$ if and only if $\theta \in \mu_1(\zeta)$, and similarly $\tau_2(\lambda) \cap \tau_2(\mu_2(\zeta)) \neq \emptyset$ if and only if $\lambda \in \mu_2(\zeta)$ for any $\zeta \in \Theta \wedge \Lambda$. From the conditional independence condition $\Theta \perp \Lambda | \Theta \wedge \Lambda$ we have

$$\tau_1(\theta) \cap \tau_2(\lambda) \cap \tau_1(\mu_1(\zeta)) = \tau_1(\theta) \cap \tau_2(\lambda) \cap \tau_2(\mu_2(\zeta)) \neq \emptyset,$$

if $\theta \in \mu_1(\zeta)$ and $\lambda \in \mu_2(\zeta)$. Therefore $\tau_1(\theta) \cap \tau_2(\lambda) \neq \emptyset$ if $\theta \in \mu_1(\eta)$ or if $\lambda \in \mu_2(\eta)$. This condition can be expressed in two ways:

- a) If $t_{\Theta \wedge \Lambda}(\theta) = t_{\Theta \wedge \Lambda}(\lambda)$, then there exists an atom $\eta \in \Theta \vee \Lambda$ such that $t_{\Theta}(\eta) = \theta$ and $t_{\Lambda}(\eta) = \lambda$.
- b) If Θ , Λ and $\Theta \wedge \Lambda$ are considered as partitions of $\Theta \vee \Lambda$, then if θ and λ are in the same block of $\Theta \wedge \Lambda$, there is an atom $\zeta \in \Theta \vee \Lambda$ such that θ and ζ are in the same block of Θ and λ and ζ are in the same block of Λ .

Note that the last condition is nothing else than the statement that the partitions in $\Theta \vee \Lambda$ associated with the coarsenings Θ and Λ commute, so that the associated saturation operators commute to (see Section 2.2). Sublattices of the partition lattice $\text{part}(U)$ of some universe satisfying condition b) are also called partition lattices of type I (Grätzer, 1978) (recall, that our order is the inverse of the order usually considered between partitions). In particular, multivariate models satisfy this condition.

We consider now Λ -valuations on such particular f.c.f. But we do not more assume that addition is idempotent nor do we necessarily assume null or unit elements in the semiring $(A; +, \times)$. Instead of a general transport operation t for Λ -valuations, we consider only projection, that is partial transport operations $\pi_{\Lambda} = t_{\Lambda} : \Phi_{\Theta} \rightarrow \Phi_{\Lambda}$, defined only for frames $\Lambda \leq \Theta$. We define Labeling and Combination as before, such that we have

1. *Labeling*: $d(\phi) = \Theta$ if ϕ is an Λ -valuation on Θ .
2. *Combination*: If $d(\phi) = \Theta$ and $d(\psi) = \Lambda$, then for $\zeta \in \Theta \vee \Lambda$, an Λ -valuation $\phi \cdot \psi$ is defined by

$$\phi \cdot \psi(\zeta) = \phi(t_{\Theta}(\zeta)) \times \psi(t_{\Lambda}(\zeta)). \quad (3.20)$$

3. *Projection*: If $d(\phi) = \Theta$ and $\lambda \in \Lambda \leq \Theta$, then an Λ -valuation $\pi_{\Lambda}(\phi)$ is defined by

$$\pi_{\Lambda}(\phi)(\lambda) = \sum_{\theta \in t_{\Theta}(\lambda)} \phi(\theta). \quad (3.21)$$

We show that in this way we get a *valuation algebra* of Λ -valuations

Axiom S0 is valid by the assumption that $(\mathcal{F}; \leq)$ is a lattice. The Semigroup Axiom S1 holds as before, the definition of combination has not changed; and so holds the Labeling Axiom S2. Axiom S4 can easily be verified. The next theorem gives us the combination Axiom S5 for Λ -valuations.

Theorem 3.10 *Let $(\mathcal{F}; \leq)$ be a lattice such that $\Theta \perp \Lambda | \Theta \wedge \Lambda$ for all frames $\Theta, \Lambda \in \mathcal{F}$. Then, if $d(\phi) = \Theta$ and $d(\psi) = \Lambda$,*

$$\pi_{\Theta}(\phi \cdot \psi) = \phi \cdot \pi_{\Theta \wedge \Lambda}(\psi).$$

Proof. The condition $\Theta \perp \Lambda | \Theta \wedge \Lambda$ implies $t_\Lambda(\theta) = t_\Lambda(\theta)(t_{\Theta \wedge \Lambda}(\theta))$ for all $\theta \in \Theta$ (Theorem 3.4). Since we have also $\Theta \perp \Lambda | \Theta \vee \Lambda$, we find $t_\Lambda(\theta) = (t_\Lambda(t_{\Theta \vee \Lambda}(\theta)))'$, hence

$$t_\Lambda(t_{\Theta \vee \Lambda}(\theta)) = t_\Lambda(t_{\Theta \wedge \Lambda}(\theta))$$

To simplify writing, it is convenient to define for $d(\phi) = \Theta$ and $S \subseteq \Theta$,

$$\phi(S) = \sum_{\theta \in S} \phi(\theta)$$

Then projection can be written as $\pi_\Lambda(\phi)(\lambda) = \phi(t_\Theta(\lambda))$. With this notation, from the definition of projection and combination, we obtain for $\theta \in \Theta$,

$$\begin{aligned} \pi_\Theta(\phi \cdot \psi)(\theta) &= \phi \cdot \psi(t_{\Theta \vee \Lambda}(\theta)) = \phi(\theta) \times \psi(t_\Lambda(t_{\Theta \vee \Lambda}(\theta))) \\ &= \phi(\theta) \times \psi(t_\Lambda(t_{\Theta \wedge \Lambda}(\theta))) = \phi(\theta) \times t_{\Theta \wedge \Lambda}(\psi)(t_{\Theta \wedge \Lambda}(\theta)) \\ &= \phi \cdot \pi_{\Theta \wedge \Lambda}(\psi)(\theta). \end{aligned}$$

This shows that $\pi_\Theta(\phi \cdot \psi) = \phi \cdot \pi_{\Theta \wedge \Lambda}(\psi)$. \square

So, in this case all axioms of a valuation algebra are satisfied, with the exception of Axiom S3 concerning unit and null valuations. So, we have the following theorem:

Theorem 3.11 *Let $(\mathcal{F}; \leq)$ be a lattice such that $\Theta \perp \Lambda | \Theta \wedge \Lambda$ holds for all frames $\Theta, \Lambda \in \mathcal{F}$. Then $(\Phi, \mathcal{F}; \leq, d, \cdot, \pi)$ is a valuation algebra (possibly without Axiom S3).*

If the semiring A has null and unit elements, then under some additional conditions, Axiom S3 of a valuation algebra in the old sense or parts of it may be satisfied. If the semiring A has a null and unit element, then Theorem 3.6 still holds for projections. If addition is idempotent, then $\pi_\Lambda(1_\Theta) = 1_\Lambda$, the valuation algebra is stable. As shown in Section 3.2, a generalised information algebra can then be derived from the valuation algebra. If the semiring A is also positive, then $\pi_\Lambda(\phi) = 0_\Lambda$ implies $\phi = 0_\Theta$, if $d(\phi) = \Theta$. In these cases Axiom S3 of the original information algebra is valid too.

Here follow a few examples of valuation algebras induced by a semiring.

Example 3.11 *Probability Potentials:* The arithmetic semiring $(\mathbb{R}^+ \cup \{0\}; +, \times)$ gives rise to the semiring of mappings of frames Θ to nonnegative real numbers. Note that addition is not idempotent, so there is no generalised information algebra relative to this semiring. This semiring valuation algebra is usually considered in the framework to Bayesian networks (Lauritzen & Spiegelhalter, 1988; Shenoy & Shafer, 1990a; Shafer, 1996). That

is why the A-valuations are called probability potentials, especially, since a nonnegative function may also be transformed into a probability distribution by normalisation. As with densities, combination of probability potentials in the valuation algebra does not directly correspond to an operation of classical probability. For an interpretation of this operation in probabilistic argumentation systems. \ominus

Example 3.12 *Subsets, Fuzzy Sets or Possibilistic Constraints:* The examples of set algebras, fuzzy sets and possibilistic constraints based on t-norms exist also as valuation algebras. \ominus

To conclude, we see that many important information or valuation algebras are induced by a semiring. We remark further, that the concept of semiring-valuation algebras may be extended to set-based semiring valuation algebras (Pouly & Kohlas, 2011) which cover examples of valuation algebras which are not semiring valuation algebras, like belief functions and generalisations thereof.

Chapter 4

Local Computation

4.1 Markov Trees

In this section we introduce a number of conditional independence structures like Markov trees, hypertrees and join trees which play an important role in local computation. These structures are well known and frequently used in multivariate models, and there they are all equivalent. Here however, we study them in the context of quasi-separoids. And in this context they are all different. In Section 4.2 we discuss how different algorithms of local computation can be defined on these different structures.

To start, we extend the notion of conditional independence to a family of domains in a join-semilattice (D, \leq) .

Definition 4.1 *Let $(D; \leq, \perp)$ be a quasi-separoid, and x_1, \dots, x_n and z elements of D , $n \geq 2$. The family $\{x_1, \dots, x_n\}$ is called conditionally independent given z , if for all disjoint subsets J and K of the index set $\{1, \dots, n\}$*

$$\bigvee_{j \in J} x_j \perp \bigvee_{k \in K} x_k \mid z. \quad (4.1)$$

Then we write $\perp\{x_1, \dots, x_n\} \mid z$.

By convention, for all $x \in D$, we define $\perp\{x\} \mid z$ and $\perp\emptyset \mid z$. Note that due to condition C3 of a q-separoid, we may assume that $J \cup K = \{1, \dots, n\}$ in this definition.

Theorem 4.1 *Assume $\perp\{x_1, \dots, x_n\} \mid z$. Then,*

1. *if σ is a permutation of $1, \dots, n$, then $\perp\{x_{\sigma(1)}, \dots, x_{\sigma(n)}\} \mid z$,*
2. *if $J \subseteq \{1, \dots, n\}$, then $\perp\{x_j : j \in J\} \mid z$,*
3. *if $y \leq x_1$, then $\perp\{y, x_2, \dots, x_n\} \mid z$,*
4. *$\perp\{x_1 \vee x_2, x_3, \dots, x_n\} \mid z$,*

$$5. \perp\{x_1 \vee z, x_2, \dots, x_n\}|z.$$

Proof. Items 1.), 2.) and 4.) are immediate consequences of the definition. Item 3.) follows from C3 and 5.) from C4. \square

In case $(D; \leq)$ is a lattice, $\perp_L\{x_1, \dots, x_n\}|z$ implies $x_1 \perp_L x_2|z$, $x_2 \perp_L x_3|z$, etc. which means that $(x_1 \vee z) \wedge (x_2 \vee z) = z$, $(x_2 \vee z) \wedge (x_3 \vee z) = z$, etc. and this implies

$$(x_1 \vee z) \wedge (x_2 \vee z) \wedge \dots \wedge (x_n \vee z) = z.$$

If the lattice $(D; \leq)$ is distributive, then

$$(\vee_{j \in J} x_j \vee z) \wedge (\vee_{k \in K} x_k \vee z) = \vee_{j \in J, k \in K} (x_j \wedge x_k) \vee z = z,$$

hence $x_j \wedge x_k \leq z$ for all $j \neq k$. Therefore, in this case $\perp_L\{x_1, \dots, x_n\}|z$ holds if and only if $x_j \perp_L x_k|z$ for all pairs of distinct j and k ; $j, k = 1, \dots, n$.

Axiom A5 can be extended to a family of conditionally independent domains.

Theorem 4.2 Assume $\perp\{x_1, \dots, x_n\}|z$ and $\phi = \phi_1 \dots \phi_n$ with $d(\phi_i) = x_i$ for $i = 1, \dots, n$. Then

$$t_z(\phi) = t_z(\phi_1) \dots t_z(\phi_n). \quad (4.2)$$

Proof. The proof is by induction. The theorem is valid for $n = 2$ by A5. Assume it holds for $n-1$. From $\perp\{x_1, \dots, x_n\}|z$ it follows that $\vee_{i=1}^{n-1} x_i \perp x_n|z$. From A5 we obtain then

$$t_z(\phi) = t_z(\phi_1 \dots \phi_{n-1}) \cdot t_z(\phi_n).$$

Using the assumption of induction $t_z(\phi_1 \dots \phi_{n-1}) = t_z(\phi_1) \dots t_z(\phi_{n-1})$ the theorem follows. \square

Let now $(D; \leq, \perp)$ be any q-separoid. Consider a tree $T = (V, E)$, with nodes V (a finite set) and edges $E \subseteq V^2$, where V^2 is the family of two-elements subsets of V . Let $\lambda : V \rightarrow D$ be a labeling of the nodes of T with domains. The pair (T, λ) is called a labeled tree. By $ne(v)$ we denote the set of neighbours of a node v , that is, the set $\{w \in V : \{v, w\} \in E\}$. For any subset U of nodes, let

$$\lambda(U) = \vee_{v \in U} \lambda(v).$$

When a node v is eliminated together with all edges $\{v, w\}$ incident to v , then a family of subtrees $\{T_{v,w} = (V_{v,w}, E_{v,w}) : w \in ne(v)\}$ of T are created, where $T_{v,w}$ is the subtree containing node $w \in ne(v)$. This allows to define the concept of a Markov tree.

Definition 4.2 *Markov Tree:* Let $(D; \leq, \perp)$ be a quasi-separoid. A labeled tree (T, λ) with $T = (V, E)$, $\lambda : V \rightarrow D$, is called a Markov tree, if for all $v \in V$,

$$\perp \{\lambda(V_{v,w}) : w \in ne(v)\} | \lambda(v). \quad (4.3)$$

Markov trees have early been identified as important independence structures for efficient computation with belief functions, using Dempsters rule (Shafer *et al.*, 1987; Shenoy & Shafer, 1990b; Kohlas & Monney, 1995). In the first of these references, conditional independence and Markov trees are studied for partition lattices, whereas in the second the multivariate model and in the third families of compatible frames are used. The concept is generalised and adapted from the probabilistic framework of Markov fields. We prove two fundamental theorems, whose proofs are adapted from (Kohlas & Monney, 1995).

The first theorem states that there is conditional independence between the domains of a node v and a subtree $V_{v,w}$ given the domain of the neighbour w .

Theorem 4.3 *If (T, λ) is a Markov tree, then, for any node v and all nodes $w \in ne(v)$,*

$$\lambda(v) \perp \lambda(V_{v,w}) | \lambda(w). \quad (4.4)$$

Proof. For a node $w \in ne(v)$, the Markov property (4.3) reads

$$\perp \{\lambda(V_{w,u}) : u \in ne(w)\} | \lambda(w).$$

Then,

$$\lambda(V_{w,v}) \perp \bigvee_{u \in ne(w) - \{v\}} \lambda(V_{w,u}) | \lambda(w). \quad (4.5)$$

Note that

$$\lambda(V_{v,w}) = \bigvee_{u \in ne(w) - \{v\}} \lambda(V_{w,u}) \vee \lambda(w).$$

Hence from C4 we obtain

$$\lambda(V_{w,v}) \perp \lambda(V_{v,w}) | \lambda(w).$$

Finally, since $\lambda(v) \leq \lambda(V_{w,v})$, we conclude (4.4) using C3. \square

This theorem, as well as the next one which states that any subtree of a Markov tree is still a Markov tree, is important for local computation schemes (see Section 4.2)

Theorem 4.4 *If (T, λ) is a Markov tree, then every subtree is also a Markov tree.*

Proof. Assume $T' = (V', E')$ to be a subtree of $T = (V, E)$ and λ' the restriction of λ to V' . Consider a node $v \in V'$ and let $ne'(v)$ be the set of its neighbours in T' . Also consider the subtrees $T'_{v,w} = (V'_{v,w}, E'_{v,w})$ in the subtree T' obtained after removing the node v and the edges incident to v . Note that $ne'(v) \subseteq ne(v)$ and $V'_{v,w} \subseteq V_{v,w}$, so that $\lambda'(V'_{v,w}) \leq \lambda(V_{v,w})$ for all $w \in ne'(v)$. Therefore, from items 2 and 3 of Theorem 4.1 we conclude that

$$\perp \lambda'(V'_{v,w} : w \in ne'(v)) | \lambda'(v)$$

for all $v \in V'$. This shows that (T', λ') is a Markov tree. \square

From Markov trees two important derived structures may be obtained. In a tree T , we may always select any node v and then number the nodes $i : V \rightarrow \{1, \dots, n\}$ if $|V| = n$, such the number i of node w is smaller than the number of any node u on the path from w to v . Assume then that in a Markov tree (T, λ) , the nodes are numbered in such a way and let $x_i = \lambda(v_i)$. To simplify, we denote the nodes in the sequel by their number. Then, for all $i = 1, \dots, n-1$ the set of nodes $\{i+1, \dots, n\}$, together with all the edges from E between these nodes, determines a subtree of T . In fact, a path in T from $j > i$ to n can not pass through any node $h \leq i$. So, the subgraph determined by the nodes $\{i+1, \dots, n\}$ is connected, hence a tree. There is exactly one node $j \in ne(i)$ such that $i < j$. Denote this node by $b(i)$. By Theorem 4.3 we have

$$x_i \perp \bigvee_{j=i+1}^n x_j | x_{b(i)}. \quad (4.6)$$

This relation is defining a hypertree according to the following definition.

Definition 4.3 *Hypertree: Let $(D; \leq, \perp)$ be a quasi-separoid. A n -element subset S of D is called a hypertree, if there is a numbering of its elements $S = \{x_1, \dots, x_n\}$ such that for all $i = 1, \dots, n-1$ there is an element $b(i) > i$ in the numbering so that*

$$x_i \perp \bigvee_{j=i+1}^n x_j | x_{b(i)}. \quad (4.7)$$

In the literature, a hypergraph is usually defined as a set of subsets; in other words a set of elements of the lattice of subsets of a set. In a generalisation of this view, we take a hypergraph to be a set of elements of any join-semilattice D . The concept of a hypertree as given in Definition 4.3 is then the corresponding transcription of the usual definition of a hypertree in the context of subset lattices. Hypertrees in the usual sense were especially

studied in relational algebra, where they were called acyclic hypergraphs and shown to have some desirable properties (Beeri *et al.*, 1983; Maier, 1983). In particular, hypertrees are interesting with respect to computational complexity (Gottlob *et al.*, 1999a; Gottlob *et al.*, 1999b; Gottlob *et al.*, 2001). These papers treat all hypertrees in the multivariate framework, whereas this issue will be taken up in Sections 4.2 and 4.3 in the more general case of quasi-separoids and semilattices or lattices of domains.

So, any Markov tree determines a hypertree; in fact many hypertrees, according to the numbering selected. Following (Shenoy & Shafer, 1990b) the sequence x_1, \dots, x_n is called a (hypertree) construction sequence. A hypertree construction sequence x_1, \dots, x_n defines a tree $T = (V, E)$ with nodes $V = \{1, \dots, n\}$ and edges $E = \{\{i, b(i)\}, i = 1, \dots, n-1\}$. In fact, T is connected: If i and j are two nodes, then the node sequences $i, b(i), b(b(i)), \dots$ and $j, b(j), b(b(j)), \dots$ determine both paths from i and j to n . So there is a path between the nodes i and j . Since the number of edges is one less than the number of nodes, T is a tree. However, the labeling $i \mapsto x_i$ does not in general yield a Markov tree. To see this consider a construction sequence $\{x_1, x_2, x_3, x_4\}$ such that $x_1 \perp x_2 \vee x_3 \vee x_4 | x_4$ and $x_2 \perp x_3 \vee x_4 | x_4$. Then $S = \{x_1, x_2, x_3, x_4\}$ is a hypertree. And the construction sequence defines the tree $T = (\{1, 2, 3, 4\}, \{\{1, 4\}, \{2, 4\}, \{3, 4\}\})$. In order that the tree T with the labeling x_i be a Markov tree, we must have $\perp\{x_1, x_2, x_3\} | x_4$ and for this to be valid, for instance $x_1 \vee x_2 \perp x_3 | x_4$ must hold. But this is not necessarily guaranteed by the construction sequence. However, we shall see that if $(D; \leq)$ is a distributive lattice, then in a q-separoid $(D; \leq, \perp_L)$ any hypertree defines in the way described a Markov tree.

Let (T, λ) with $T = (V, E)$ again be a Markov tree and consider two nodes v and u . Let w be any node on the path between v and u , different from v and u , and v' and u' the neighbours of w on the path from v to w and u to w respectively. Then, from the Markov property (4.3) it follows that

$$\lambda(V_{w,v'}) \perp \lambda(V_{w,u'}) | \lambda(w)$$

and therefore, by C3 $\lambda(v) \perp \lambda(u) | \lambda(w)$. And this holds for any node w on the path between v and u . This is a defining property of another concept.

Definition 4.4 *Join Tree:* Let $(D; \leq, \perp)$ be a quasi-separoid and (T, λ) with $T = (V, E)$ a tree. If for any pair of nodes $v, u \in V$ and for any node w on the path from v to u

$$\lambda(v) \perp \lambda(u) | \lambda(w), \tag{4.8}$$

then (T, λ) is called a join tree.

Join trees have been considered in relational database theory (Beeri *et al.*, 1983; Maier, 1983) and, under varying names, also in local computation theory

for uncertainty calculi, in particular Bayesian networks, see for instance (Lauritzen & Spiegelhalter, 1988; Cowell *et al.*, 1999; Shenoy & Shafer, 1990a), but exclusively in the multivariate framework. In this case, we have $\lambda(v) \perp_L \lambda(u) | \lambda(w)$ if and only if $\lambda(v) \wedge \lambda(u) \leq \lambda(w)$. This is called the running intersection property. The present definition has been adapted from the multivariate framework.

As seen above, any Markov tree is also a join tree. However, as with hypertrees, a join tree is not a Markov tree in general. Consider the tree $T = (\{1, 2, 3, 4\}, \{\{1, 4\}, \{2, 4\}, \{3, 4\}\})$ already considered above and assume that x_i is a labeling of this tree, such that $x_1 \perp x_2 | x_4$, $x_1 \perp x_3 | x_4$ and $x_2 \perp x_3 | x_4$. The tree T with the labeling x_i is then a join tree. These pairwise conditional independence relations are however not sufficient to imply the Markov property $\perp \{x_1, x_2, x_3\} | x_4$ for the labeled tree, except if in the q-separoid $(D; \leq, \perp_L)$ the lattice D is distributive. In fact, if D is distributive, then the three concepts are equivalent in the q-separoid $(D; \leq, \perp_L)$, a fact which has been known for long in the framework of multivariate models.

Before we prove this result, we show that a hypertree in the quasi-separoid $(D; \leq, \perp_L)$ induces always a join tree. It is open whether this is true for any q-separoid $(D; \leq, \perp)$.

Theorem 4.5 *Let $(D; \leq)$ be a lattice and $(D; \leq, \perp_L)$ a quasi-separoid, and S a hypertree with construction sequence x_1, \dots, x_n . Then the labeled tree (T, λ) with $T = (V, E)$ with $V = \{1, \dots, n\}$ and $E = \{\{i, b(i)\} : i = 1, \dots, i-1\}$ and $\lambda(i) = x_i$ is a join tree.*

Proof. Consider two nodes i and j and assume $i \leq j$. Then (4.7) implies

$$x_i \wedge x_j \leq x_i \wedge (\bigvee_{k=i+1}^n x_k) = x_i \wedge x_{b(i)} \leq x_{b(i)}. \quad (4.9)$$

By iterating this argument with $x_i \wedge x_j \leq x_{b(i)} \wedge x_j \leq x_{b(b(i))}$, we see that $x_i \wedge x_j \leq x_h$ for any node h on the path from i to n as long as $h < j$. Let i_1 be the first node on this path such that $j \leq i_1$. Then we still have $x_i \wedge x_j \leq x_{i_1}$. Further, in the same way we conclude that

$$x_i \wedge x_j \leq x_{i_1} \wedge x_j \leq x_j \wedge (\bigvee_{k=j+1}^n x_k) = x_j \wedge x_{b(j)} \leq x_{b(j)}. \quad (4.10)$$

By iterating this, we obtain $x_i \wedge x_j \leq x_h$ for all h such that $i_1 \leq h < j_1$, where j_1 is the first index on the path from j to n which is greater than i_1 . Then we alternate this reasoning between the paths from i to n , starting with i_1 , with the path from j to n , starting with j_1 , until we reach the common node on both paths. Thus we conclude that $x_i \wedge x_j \leq x_h$ for all nodes on the path from i to j . Therefore (T, λ) is a join tree. \square

Now, we show the equivalence of the concepts of a Markov tree, a hypertree and a join tree in the context of a q-separoid $(D; \leq, \perp_L)$, if D is a distributive lattice.

Theorem 4.6 *Let $(D; \leq, \perp_L)$ be a quasi-separoid, D a distributive lattice and the labeled tree (T, λ) with $T = (V, E)$ a join tree. Then*

1. *The set $\lambda(V)$ is a hypertree.*
2. *The labeled tree (T, λ) is a Markov tree.*

Proof. (1) We have to find a hypertree construction sequences. For this purpose select any node $v \in V$ and let $|V| = n$. Then there is a numbering $i : V \rightarrow \{1, \dots, n\}$, such that $i(v) = n$ and $i(u) < i(w)$ if node w is on the path from node u to v . Define $x_{i(u)} = \lambda(u)$. We claim that x_1, \dots, x_n is a hypertree construction sequence and hence $\lambda(V)$ a hypertree. In order to show this, we identify the nodes by their number and define $b(i) = j$, if $i < j$ and $\{i, j\} \in E$ for $i = 1, \dots, n-1$. Note that $b(i)$ is uniquely determined, since there is only one path from i to n . Now, by distributivity

$$x_i \wedge (\bigvee_{j=i+1}^n x_j) = \bigvee_{j=i+1}^n (x_i \wedge x_j).$$

If $i < j$, the path from i to j passes through node $b(i)$, hence $x_i \wedge x_j \leq x_{b(i)}$ for all $j = i+1, \dots, n$. Therefore,

$$x_i \wedge (\bigvee_{j=i+1}^n x_j) \leq x_{b(i)}.$$

But $i+1 \leq b(i) \leq n$. So on the other hand,

$$x_i \wedge (\bigvee_{j=i+1}^n x_j) \geq x_i \wedge x_{b(i)}$$

and this implies then

$$x_i \wedge (\bigvee_{j=i+1}^n x_j) = x_i \wedge x_{b(i)}$$

In a distributive lattice, this is equivalent to $x_i \perp_L \bigvee_{j=i+1}^n x_j | x_{b(i)}$. This shows that x_1, \dots, x_n is a hypertree construction sequence.

(2) Since D is distributive, the Markov property (4.3) holds if and only if $\lambda(V_{v,w}) \perp_L \lambda(V_{v,u}) | \lambda(v)$ for all pairs of distinct neighbours w and u of v . We claim that these pairwise conditional independence relations hold in a join tree. In fact, by distributivity

$$\lambda(v) \leq (\lambda(V_{v,w}) \vee \lambda(v)) \wedge (\lambda(V_{v,u}) \vee \lambda(v)) \quad (4.11)$$

$$\begin{aligned} &= \left(\bigvee_{w' \in V_{v,w}} \lambda(w') \vee \lambda(v) \right) \wedge \left(\bigvee_{u' \in V_{v,u}} \lambda(u') \vee \lambda(v) \right) \\ &= \left(\bigvee_{w' \in V_{v,w}, u' \in V_{v,u}} (\lambda(w') \wedge \lambda(u')) \right) \\ &\wedge \left(\bigvee_{w' \in V_{v,w}} (\lambda(w') \wedge \lambda(v)) \right) \wedge \left(\bigvee_{u' \in V_{v,u}} (\lambda(u') \wedge \lambda(v)) \right) \wedge \lambda(v) \\ &\leq \lambda(v), \end{aligned}$$

since v is on all the pathes from nodes w' in $V_{v,w}$ to nodes u' in $V_{v,u}$, so that $\lambda(w') \wedge \lambda(u') \leq \lambda(v)$ by the join tree property. This shows that $\lambda(V_{v,w}) \perp_L \lambda(V_{v,u}) | \lambda(v)$, hence (T, λ) is a Markov tree. \square

In summary, any Markov tree is a join tree and induces a hypertree, but not vice versa. Moreover, in a quasi-separoid $(D; \leq, \perp_L)$ a hypertree induces a join tree, but again, not vice versa. If $(D; \leq)$ in the q-separoid $(D; \leq, \perp_L)$ however is a distributive lattice, the concepts of Markov, hyper and join tree become equivalent. A join tree is then a Markov tree and induces a hypertree and vice versa. In this case $(D; \leq, \perp_L)$ is also strong separoid (Theorem 2.4). This applies in particular to multivariate models.

4.2 Computing in Markov Trees

Generalising the local computation scheme for inference in Bayesian networks (Lauritzen & Spiegelhalter, 1988), Shenoy and Shafer proposed an axiomatic scheme for general local computation, in addition to probability propagation, especially also for belief functions (Shenoy & Shafer, 1990a). This laid the basis for valuation algebras (Kohlas & Shenoy, 2000; Kohlas, 2003a) as an axiomatic foundation of local computation. The general information algebra proposed here extend and generalise valuation algebras as structures for local computation schemes. We note that in (Shafer *et al.*, 1987) already local computation of belief on functions on partition lattices were described, which is an instance more general than valuation algebras, and in (Kohlas & Monney, 1995) local computation on families of compatible frames is discussed. The present discussion generalises both of these approaches.

Local computation schemes in the framework of information algebras propose computational solutions to the so-called projection problem. It is assumed that information is given in pieces, which must be combined, and then the part relative to some given question is to be extracted. So, let ϕ_1, \dots, ϕ_n be n pieces of information and $\phi = \phi_1 \cdot \dots \cdot \phi_n$ the aggregated information. Assume that the question to be examined is represented by some element x of D . Then the problem is to compute

$$t_x(\phi) = t_x(\phi_1 \cdot \dots \cdot \phi_n). \quad (4.12)$$

This is called the *projection problem*. Let x_i denote the domain of ϕ_i . Then the domain of ϕ equals $x_1 \vee \dots \vee x_n$ according to the labeling axiom. We may presume that the basic operations of an information algebra, combination and transport, have a degree of complexity which grows with the size of the domain and, in many cases, combination and transport may become computationally infeasible on large domains. Therefore, the naive solution of the projection problem which sequentially combines the factors ϕ_1, ϕ_2 up to ϕ_n and then extract the part relating to x by the transport operator

applied to ϕ may be infeasible or at least inefficient. In the case of Bayesian networks, (Lauritzen & Spiegelhalter, 1988) have shown that the projection problem may be solved in some circumstances by arranging the operations in such a way that they take place on the domains x_i of the factors of the combination ϕ and (Shenoy & Shafer, 1990a) have shown that this is possible more generally in the case of abstract valuation algebras. This is called *local computation*. We show here that local computation is also possible with generalised information algebras.

If local computation can be used to solve the projection problem, then the complexity is determined by the operations of combination and transport on the domains x_i . Assume some measure of complexity $c(x_i)$ depending on the domain and let $c = \max\{c(x_1), \dots, c(x_n)\}$. Then the complexity of local computation is $n \cdot c$, hence *linear* in the problem size as measured by the number of factors to be combined. The complexity measure $c(x_i)$ depends much on the instance of the information or valuation algebra. It may be polynomial or exponential in some parameter measuring the size of domains x_i , depending on the instance, see (Pouly & Kohlas, 2011). But the linearity in the problem size, once c is given, makes local computation in many cases feasible, because the structure of practical problems often guarantee small domains x_i , and therefore a reasonably small value for c . This represents a situation studied more generally in *parameterised complexity theory*. In the case of Bayesian networks, local computation is shown to be closely related to conditional independence structures, see for example (Cowell *et al.*, 1999). The same is the case in relational algebra, (Beeri *et al.*, 1983; Maier, 1983). In fact, viewed from the point of view of generalised information algebras, these structures can be identified as Markov trees and hypertrees.

In this section we assume throughout a generalised information algebra $(\Phi, D; \leq, \perp, \cdot, t)$. Assume that the domains x_i of the factors of a factorisation (4.12) of ϕ form a *Markov tree*. More precisely, let (T, λ) with $T = (V, E)$ be a Markov tree such that $|V| = n$ and each node v corresponds to exactly one domain x_i , such that $x_i = \lambda(v)$. Under this assignment denote then ϕ_i by ϕ_v , such that

$$\phi = \prod_{v \in V} \phi_v, \quad d(\phi_v) = \lambda(v). \quad (4.13)$$

We call such a factorisation a *Markov tree factorisation*. As usual $T_{v,w} = (V_{v,w}, E_{v,w})$ denotes the subtree obtained by removing node v and containing the neighbour w of v . As we know, this is still a Markov tree (Theorem 4.4). Assume further that $x = \lambda(v)$ for some node v of the Markov tree, such that $t_{\lambda(v)}(\phi)$ is to be computed. Such a projection problem has a local computation solution, as the following theorem shows.

Theorem 4.7 *Let (T, λ) be a Markov tree with $T = (V, E)$ and ϕ given by a Markov tree factorisation (4.13) according to this Markov tree. Then, for*

any node $v \in V$,

$$t_{\lambda(v)}(\phi) = \phi_v \cdot \prod_{w \in ne(v)} t_{\lambda(v)}(t_{\lambda(w)}(\phi_{v,w})), \quad (4.14)$$

where

$$\phi_{v,w} = \prod_{u \in V_{v,w}} \phi_u. \quad (4.15)$$

Proof. Note that $d(\phi_{v,w}) = \lambda(V_{v,w})$. By Theorem 4.3, $\lambda(v) \perp \lambda(V_{v,w}) | \lambda(w)$. Using axiom A4 we obtain

$$t_{\lambda(v)}(\phi_{v,w}) = t_{\lambda(v)}(t_{\lambda(w)}(\phi_{v,w})).$$

Now,

$$\phi = \phi_v \cdot \prod_{w \in ne(v)} \phi_{v,w}$$

By C1 and C2, $\lambda(v) \perp \bigvee_{w \in ne(v)} \lambda(V_{v,w}) | \lambda(v)$. So by axiom A5,

$$t_{\lambda(v)}(\phi) = t_{\lambda(v)}(\phi_v) \cdot t_{\lambda(v)}\left(\prod_{w \in ne(v)} \phi_{v,w}\right).$$

Further, the Markov property $\perp \{\lambda(V_{v,w}) : w \in ne(v)\} | \lambda(v)$ implies, using Theorem 4.2,

$$t_{\lambda(v)}\left(\prod_{w \in ne(v)} \phi_{v,w}\right) = \prod_{w \in ne(v)} t_{\lambda(v)}(\phi_{v,w}).$$

Finally, by axiom A6 we have $t_{\lambda(v)}(\phi_v) = \phi_v$, hence

$$t_{\lambda(v)}(\phi) = \phi_v \cdot \prod_{w \in ne(v)} t_{\lambda(v)}(t_{\lambda(w)}(\phi_{v,w})).$$

which concludes the proof. \square

Formulas (4.14) and (4.15) define a *recursive scheme* of local computation for the solution of the projection problem. Once the subproblems $t_{\lambda(w)}(\phi_{v,w})$ in the Markov subtree $T_{v,w}$ are solved for all neighbours w of v , only transports from node w to node v and combinations on node v have to be executed. These are *local* operations on the domain $\lambda(v)$. The anchors for the recursion are the trivial one-node Markov trees $\{u\}$, where the projection problem $t_{\lambda(u)}(\phi_u) = \phi_u$ is trivial. So this is indeed a local computation scheme.

It is possible to represent this computation scheme as a message passing method on a tree, as already shown for valuation algebras by (Shenoy & Shafer, 1990a). For two neighbouring nodes v and w define a message from w to v by

$$\mu_{w \rightarrow v} = t_{\lambda(v)}(t_{\lambda(w)}(\phi_{v,w})). \quad (4.16)$$

Select arbitrarily a node v of the tree as a root node and direct all arcs of the tree towards this node. Note that then all nodes, except the root node, have exactly one outgoing arc. Further, there is at least one leaf node without incoming arc. Each such leaf node u has a unique neighbour w to which it can send the message

$$\mu_{u \rightarrow w} = t_{\lambda(w)}(\phi_u).$$

Once a node u has received messages through all its *incoming* arcs, it can compute the message for the unique outgoing arc and send it to its neighbour w by

$$\mu_{u \rightarrow w} = t_{\lambda(w)}(\phi_u \cdot \prod_{n \in ne(u) - \{w\}} \mu_{n \rightarrow u}).$$

Once the root node v has received all its messages through this procedure, it can compute the solution of the projection problem according to (4.14) by

$$t_{\lambda(v)}(\phi) = \phi_v \cdot \prod_{w \in ne(v)} \mu_{w \rightarrow v}.$$

This message passing scheme is also called a *collect algorithm*.

What is even more, the root node can now send messages back to all its neighbours. If in the collect phase the messages have been cached, all neighbours w of v can then compute the projection $t_{\lambda(w)}(\phi)$,

$$t_{\lambda(w)}(\phi) = \phi_w \cdot \prod_{u \in ne(w)} \mu_{u \rightarrow w}.$$

These nodes are then in a position to send themselves messages back to all their neighbours through their incoming arcs, and so on. Finally, by sending messages in this way backwards towards the leafs, at the end $t_{\lambda(w)}(\phi)$ has been computed for all nodes w of the Markov tree. This second phase of computation is also called a *distribute algorithm*.

The formulae of this local computation scheme simplify somewhat, if one computes in a valuation algebra. Then, using the formula for transport in a valuation algebra (3.8), we obtain for the messages

$$\mu_{u \rightarrow w} = \pi_{\lambda(u) \wedge \lambda(w)}(\phi_u \cdot \prod_{n \in ne(u) - \{w\}} \mu_{n \rightarrow u}).$$

and

$$\pi_{\lambda(v)}(\phi) = \phi_v \cdot \prod_{w \in ne(v)} \mu_{w \rightarrow v}.$$

This is the version of the collect algorithm, usually defined in the multivariate setting (Shafer & Shenoy, 1990; Kohlas & Shenoy, 2000; Pouly & Kohlas, 2011). Many combinations of messages are computed several times. There are ways to reduce these redundant computations by creating binary trees or using division (which is possible in some cases, see (Kohlas, 2003a)). In the multivariate framework other variants of local computation are possible, like fusion or bucket elimination schemes, based on successive variable eliminations (Shenoy, 1992; Dechter, 1999), methods which are not available in our more general setting.

4.3 Computation in Hypertrees

Local computation can also be defined on a hypertree. Suppose that the domains x_1, \dots, x_n in a projection problem

$$\phi = \phi_1 \cdot \dots \cdot \phi_n, \quad d(\phi_i) = x_i,$$

define a hypertree construction sequence and that $x = x_n$ in the projection problem (4.12). Then a local computation solution can be obtained as follows: Define

$$y_i = x_{i+1} \vee \dots \vee x_n, \quad i = 1, \dots, n-1.$$

First, we eliminate the first domain x_1 in the sequence by computing $t_{y_1}(\phi)$, using the hypertree condition (Definition 4.3). From $x_1 \perp_{y_1} y_1$ and axioms A2, A5 and A6 we obtain,

$$t_{y_1}(\phi) = t_{y_1}(\phi_1) \cdot t_{y_1}(\phi_2 \cdot \dots \cdot \phi_n) = t_{y_1}(\phi_1) \cdot \phi_2 \cdot \dots \cdot \phi_n.$$

The hypertree condition $x_1 \perp_{y_1} x_{b(1)}$, (see (4.7), implies $t_{y_1}(\phi_1) = t_{y_1}(t_{x_{b(1)}}(\phi_1))$ and therefore,

$$t_{y_1}(\phi) = t_{y_1}(t_{x_{b(1)}}(\phi_1)) \cdot \phi_2 \cdot \dots \cdot \phi_n.$$

Since $x_{b(1)} \leq y_1 = d(\phi_2 \cdot \dots \cdot \phi_n)$ we conclude (see Lemma 3.1, 5.) that

$$t_{y_1}(\phi) = t_{y_1}(t_{x_{b(1)}}(\phi_1)) \cdot t_{y_1}(\phi_2 \cdot \dots \cdot \phi_n) = t_{x_{b(1)}}(\phi_1) \cdot \phi_2 \cdot \dots \cdot \phi_n.$$

Define $\psi_i^1 = \phi_i$ and then $\psi_{b(1)}^2 = \psi_{b(1)}^1 \cdot t_{b(1)}(\psi_1^1)$ and $\psi_j^2 = \psi_j^1$ for $j = 2, \dots, n$ and $j \neq b(1)$. Note that $d(\psi_j^2) = x_j$ for $j = 2, \dots, n$. Then after elimination of domain x_1 we obtain a new factorisation

$$t_{y_1}(\phi) = \psi_2^2 \cdot \dots \cdot \psi_n^2.$$

We may now proceed in the same way, eliminating domains x_2, x_3, \dots . By induction let's assume

$$t_{y_{i-1}}(\phi) = \psi_i^i \cdot \dots \cdot \psi_n^i, \quad d(\psi_j^i) = x_j, j = i, \dots, n. \quad (4.17)$$

Since $y_i \leq y_{i-1}$ we have $t_{y_i}(t_{y_{i-1}}(\phi)) = t_{y_i}(\phi)$. Now we eliminate domain x_i in y_{i-1} in (4.17) in the same way as we did above for x_1 and obtain

$$\begin{aligned} t_{y_i}(\phi) &= t_{y_i}(\psi_i^i \cdot \dots \cdot \psi_n^i) \\ &= t_{y_i}(t_{x_{b(i)}}(\psi_i^i)) \cdot \psi_{i+1}^i \cdot \dots \cdot \psi_n^i \\ &= t_{x_{b(i)}}(\psi_i^i) \cdot \psi_{i+1}^i \cdot \dots \cdot \psi_n^i. \end{aligned}$$

Define

$$\psi_{b(i)}^{i+1} = \psi_{b(i)}^i \cdot t_{x_{b(i)}}(\psi_i^i) \quad (4.18)$$

and $\psi_j^{i+1} = \psi_j^i$ for $j = i+1, \dots, n$ and $j \neq b(i)$. We still have $d(\psi_j^{i+1}) = x_j$ for $j = i+1, \dots, n$. Thus we obtain the new factorisation

$$t_{y_i}(\phi) = \psi_{i+1}^{i+1} \cdot \dots \cdot \psi_n^{i+1}, \quad d(\psi_j^{i+1}) = x_j, j = i+1, \dots, n.$$

This concludes the induction. At the end, for $i = n$, we obtain in this way

$$t_{x_n}(\phi) = \psi_n^n. \quad (4.19)$$

This solves the projection problem on the hypertree $\{x_1, \dots, x_n\}$ and does it by local computation: In every step $i = 1, \dots, n-1$ a transport operation $t_{x_{b(i)}}(\psi_i^i)$ and a combination operation of the result with $\psi_{b(i)}^i$ have to be executed and this $n-1$ times. These are all local operations on domains $x_{b(i)}$. So, here we have a second local computation scheme, this time on a hypertree.

Note that a Markov tree induces a hypertree, even many hypertrees. In this case, it can be seen that the hypertree computation scheme just described essentially corresponds to the collect algorithm in the Markov tree. However, since a hypertree does not necessarily induce a Markov tree, the backwards distribute algorithm is not available in general in a hypertree. If the generalised information algebra however is *idempotent*, then the distribute algorithm gives the correct result for hypertrees too. This is formulated in the following theorem

Theorem 4.8 *Assume $\{x_1, \dots, x_n\}$ to be a hypertree with a hypertree construction sequence x_1, \dots, x_n , $\phi = \phi_1 \cdot \dots \cdot \phi_n$ with $d(\phi_i) = x_i$ and ψ_i^i the intermediate results computed in the collect algorithm in this sequence. Define*

$$\mu_{b(i) \rightarrow i} = t_{x_i}(t_{x_{b(i)}}(\phi)).$$

Then, for $i = n - 1, \dots, 1$, if axiom A7 (Idempotency) holds,

$$t_{x_i}(\phi) = \mu_{b(i) \rightarrow i} \cdot \psi_i^i. \quad (4.20)$$

Proof. Define as before $y_i = x_{i+1} \vee \dots \vee x_n$. Then, according to the collect algorithm above

$$t_{y_i}(\phi) = \psi_{i+1}^{i+1} \cdot \dots \cdot \psi_n^{i+1}, \quad d(\psi_j^{i+1}) = x_j, j = i + 1, \dots, n. \quad (4.21)$$

Since $x_{b(i)} \leq y_i$, we obtain

$$\psi_i^i \cdot \mu_{b(i) \rightarrow i} = \psi_i^i \cdot t_{x_i}(t_{x_{b(i)}}(\phi)) = \psi_i^i \cdot t_{x_i}(t_{x_{b(i)}}(t_{y_i}(\phi))).$$

Further, as x_1, \dots, x_n is a hypertree construction sequence, we have $x_i \perp y_i | x_{b(i)}$ or $y_i \perp x_i | x_{b(i)}$ (C1). Apply axiom A4 and (4.21) to obtain

$$\psi_i^i \cdot \mu_{b(i) \rightarrow i} = \psi_i^i \cdot t_{x_i}(t_{y_i}(\phi)) = \psi_i^i \cdot t_{x_i}(\psi_{i+1}^{i+1} \cdot \dots \cdot \psi_n^{i+1}).$$

Using (4.18) and axiom A5 with $x_i \perp y_i | x_i$ we obtain further

$$\psi_i^i \cdot \mu_{b(i) \rightarrow i} = t_{x_i}(\psi_i^i \cdot \psi_{i+1}^i \cdot \dots \cdot \psi_n^i \cdot t_{x_{b(i)}}(\psi_i^i)).$$

Now we use idempotency to show that

$$\psi_i^i \cdot \psi_{i+1}^i \cdot \dots \cdot \psi_n^i \cdot t_{x_{b(i)}}(\psi_i^i) = \psi_i^i \cdot \psi_{i+1}^i \cdot \dots \cdot \psi_n^i$$

In fact, since we assume idempotency, Lemma 3.3 applies, and so do Lemma 3.1, 5.) and 2.),

$$\begin{aligned} & \psi_i^i \cdot \psi_{i+1}^i \cdot \dots \cdot \psi_n^i \cdot t_{x_{b(i)}}(\psi_i^i) \\ &= t_{x_i \vee x_{b(i)}}(\psi_i^i) \cdot \prod_{j=i+1}^n \psi_j^i = t_{y_{i-1}}(t_{x_i \vee x_{b(i)}}(\psi_i^i)) \cdot \prod_{j=i+1}^n t_{y_{i-1}}(\psi_j^i) \\ &= t_{y_{i-1}}(\psi_i^i) \cdot \prod_{j=i+1}^n t_{y_{i-1}}(\psi_j^i) = \prod_{j=i}^n t_{y_{i-1}}(\psi_j^i) \\ &= \prod_{j=i}^n \psi_j^i. \end{aligned}$$

Then we obtain finally

$$\psi_i^i \cdot \mu_{b(i) \rightarrow i} = t_{x_i}(\psi_i^i \cdot \dots \cdot \psi_n^i) = t_{x_i}(t_{y_{i-1}}(\phi)) = t_{x_i}(\phi),$$

since $x_i \leq y_{i-1}$. This concludes the proof. \square

According to this theorem, once $t_{x_n}(\phi)$ has been computed in the n -th step of the collect algorithm, the other projection problems $t_{x_i}(\phi)$ can be computed in the inverse order $i = n - 1, \dots, 1$ of the construction sequence.

At step i , $t_{x_j}(\phi)$ is known for all $j \geq i$, and then (4.20) allows to compute $t_{x_{i-1}}(\phi)$, since $b(i-1) \geq i$.

The assumptions that the domains of the factors of a projection problem form just a Markov tree or a hypertree is of course very strong. But as usual, the existence of unit elements allows the use of covering Markov trees or hypertrees. This is in generalised information algebras just as in valuation algebras (Kohlas, 2003a; Pouly & Kohlas, 2011). In the multivariate case, covering Markov trees may be found by sequences of variable eliminations. Such procedures are not available in the more general case of general q-separoids $(D; \leq \perp)$. So, finding good covering Markov trees is an open problem. Also, like in valuation algebras the semigroup (Φ, \cdot) may allow for division as a partially inverse operation of combination, (Lauritzen & Jensen, 1997; Kohlas, 2003a). This will be not discussed here, we refer to (Kohlas, 2003a; Kohlas & Wilson, 2006) in the case of valuation algebras.

Part II

Domain-Free Algebras

Chapter 5

Domain-Free Information Algebras

5.1 Unlabeling of Information

In a labeled information algebra $(\Phi, D; \leq, \perp, d, \cdot, t)$ different pieces of information ϕ and ψ with different labels $d(\phi) = x$ and $d(\psi) = y$ may describe the same information, namely, if

$$t_{x \vee y}(\phi) = t_{x \vee y}(\psi). \quad (5.1)$$

In fact, then we see that

$$\phi = t_x(t_{x \vee y}(\phi)) = t_x(t_{x \vee y}(\psi)) = t_x(\psi),$$

and, similarly,

$$\psi = t_y(\phi).$$

So, ϕ and ψ really describe the same information, only once with respect to domain x and once with respect to domain y . This motivates to define the relation

$$\phi \equiv_{\sigma} \psi, \text{ if } d(\phi) = x, d(\psi) = y \text{ and } t_{x \vee y}(\phi) = t_{x \vee y}(\psi). \quad (5.2)$$

This relation can be defined alternatively according to the following lemma.

Lemma 5.1 *The relation $\phi \equiv_{\sigma} \psi$ holds if and only if $t_z(\phi) = t_z(\psi)$ for any $z \in D$.*

Proof. If $t_z(\phi) = t_z(\psi)$ and $d(\phi) = x$, $d(\psi) = y$, then, for $z = x \vee y$, $t_{x \vee y}(\phi) = t_{x \vee y}(\psi)$, hence $\phi \equiv_{\sigma} \psi$.

Conversely assume $\phi \equiv_\sigma \psi$, that is, $t_{x \vee y}(\phi) = t_{x \vee y}(\psi)$. Then we have also $t_{x \vee y \vee z}(\phi) = t_{x \vee y \vee z}(\psi)$. Further, $t_z(t_{x \vee y \vee z}(\phi)) = t_z(\phi)$ and the same for ψ , hence indeed $t_z(\phi) = t_z(\psi)$. \square

The relation \equiv_σ is clearly an equivalence relation in Φ . It is even a congruence relative to the operation of combination and transport as the following theorem shows.

Theorem 5.1 *The relation \equiv_σ is a congruence relative to combination and transport in the labeled information algebra $(\Phi, D; \leq, \perp, d, \cdot, t)$.*

Proof. Assume $\phi_1 \equiv_\sigma \phi_2$ and $d(\phi_1) = x$, $d(\phi_2) = y$. Consider an element $\psi \in \Phi$ with $d(\psi) = z$. We show that $\phi_1 \cdot \psi \equiv_\sigma \phi_2 \cdot \psi$. From $z \perp x \vee y \vee z \mid x \vee y \vee z$ (C1) it follows that $x \perp z \mid x \vee y \vee z$ (C2 and C3). Therefore, by axiom A5

$$t_{x \vee y \vee z}(\phi_1 \cdot \psi) = t_{x \vee y \vee z}(\phi_1) \cdot t_{x \vee y \vee z}(\psi).$$

In the same way, $y \perp z \mid x \vee y \vee z$, and

$$t_{x \vee y \vee z}(\phi_2 \cdot \psi) = t_{x \vee y \vee z}(\phi_2) \cdot t_{x \vee y \vee z}(\psi).$$

Then $\phi_1 \equiv_\sigma \phi_2$ implies, using Lemma 5.1,

$$t_{x \vee y \vee z}(\phi_1 \cdot \psi) = t_{x \vee y \vee z}(\phi_2 \cdot \psi).$$

But this means that $\phi_1 \cdot \psi \equiv_\sigma \phi_2 \cdot \psi$.

Also $\phi \equiv_\sigma \psi$ implies $t_z(\phi) = t_z(\psi)$, hence $t_z(\phi) \equiv_\sigma t_z(\psi)$. This proves congruence relative to transport. \square

This congruence allows us to consider the quotient structure Φ/σ , consisting of the equivalence classes $[\phi]_\sigma$ of the equivalence relation \equiv_σ . Combination and transport are well-defined in this structure by

$$\begin{aligned} [\phi]_\sigma \cdot [\psi]_\sigma &= [\phi \cdot \psi]_\sigma, \\ \epsilon_x([\phi]_\sigma) &= [t_x(\phi)]_\sigma. \end{aligned}$$

It is evident that combination in Φ/σ is associative and commutative, since it is so in Φ . Further, the classes $[1_x]_\sigma$ and $[0_x]_\sigma$ are the unit and null elements of combination in Φ/σ . So, $(\Phi/\sigma, \cdot)$ is a commutative semigroup with unit and null element. In addition, if $d(\phi) = x$, then $\epsilon_x([\phi]_\sigma) = [\phi]_\sigma$, hence, in particular, $\epsilon_x([1_x]_\sigma) = [1_x]_\sigma$ and $\epsilon_x([\phi]_\sigma) = [0_x]_\sigma$ if and only if $[\phi]_\sigma = [0_x]_\sigma$.

The following theorem shows how the axioms A4 A5 and A7 of the labeled information algebra $(\Phi, D; \leq, \perp, d, \cdot, t)$ translate into this new algebraic structure.

Theorem 5.2 *Let $(\Phi, D; \leq, \perp, d, \cdot, t)$ be a labeled generalised information algebra. Then in Φ/σ the following holds:*

1. $\epsilon_x([\phi]_\sigma) = [\phi]_\sigma$ and $x \perp y | z$ imply $\epsilon_y([\phi]_\sigma) = \epsilon_y(\epsilon_z([\phi]_\sigma))$.
2. $\epsilon_x([\phi]_\sigma) = [\phi]_\sigma$, $\epsilon_y([\psi]_\sigma) = [\psi]_\sigma$ and $x \perp y | z$ imply $\epsilon_z([\phi]_\sigma \cdot [\psi]_\sigma) = \epsilon_z([\phi]_\sigma) \cdot \epsilon_z([\psi]_\sigma)$.
3. If, in addition, Idempotency A7 holds, then $\epsilon_x([\phi]_\sigma) \cdot [\phi]_\sigma = [\phi]_\sigma$.

Modell Proof. 1.) We have by definition and assumption $\epsilon_x([\phi]_\sigma) = [t_x(\phi)]_\sigma = [\phi]_\sigma$. We may therefore select a representant of the class $[\phi]_\sigma$ such that $t_x(\phi) = \psi$, hence $d(\psi) = x$. Then by axiom A4, $x \perp y | z$ implies $t_y(\psi) = t_y(t_z(\psi))$, Therefore, we obtain

$$\epsilon_y([\psi]_\sigma) = [t_y(\psi)]_\sigma = [t_y(t_z(\psi))]_\sigma = \epsilon_y(\epsilon_z([\psi]_\sigma)).$$

Since $[\psi]_\sigma = [\phi]_\sigma$, we have $\epsilon_y([\phi]_\sigma) = \epsilon_y(\epsilon_z([\phi]_\sigma))$, as claimed.

2.) As above, we have $\epsilon_x([\phi]_\sigma) = [t_x(\phi)]_\sigma = [\phi]_\sigma$ and $\epsilon_y([\psi]_\sigma) = [t_y(\psi)]_\sigma = [\psi]_\sigma$. Then $x \perp y | z$ implies according to axiom A5,

$$t_z(t_x(\phi) \cdot t_y(\psi)) = t_z(t_x(\phi)) \cdot t_z(t_y(\psi)).$$

Thus, we have

$$\begin{aligned} \epsilon_z([\phi]_\sigma \cdot [\psi]_\sigma) &= \epsilon_z([t_x(\phi)]_\sigma \cdot [t_y(\psi)]_\sigma) = \epsilon_z([t_x(\phi) \cdot t_y(\psi)]_\sigma) \\ &= [t_z(t_x(\phi) \cdot t_y(\psi))]_\sigma = [t_z(t_x(\phi)) \cdot t_z(t_y(\psi))]_\sigma \\ &= [t_z(t_x(\phi))]_\sigma \cdot [t_z(t_y(\psi))]_\sigma = \epsilon_z([t_x(\phi)]_\sigma) \cdot \epsilon_z([t_y(\psi)]_\sigma) \\ &= \epsilon_z([\phi]_\sigma) \cdot \epsilon_z([\psi]_\sigma). \end{aligned}$$

This verifies item 2 of the the theorem.

3.) We have $\epsilon_x([\phi]_\sigma) \cdot [\phi]_\sigma = [t_x(\phi) \cdot \phi]_\sigma$. Assume that $d(\phi) = y$. By Lemma 3.3, $\phi \cdot t_x(\phi) = t_{x \vee y}(\phi)$. But $\phi \equiv_\sigma t_{x \vee y}(\phi)$, hence indeed $\epsilon_x([\phi]_\sigma) \cdot [\phi]_\sigma = [\phi]_\sigma$. \square

The quotient structure considered here amounts in fact to unlable the pieces of information ϕ and to obtain a domain-free representation of pieces of information $[\phi]_\sigma$. This will be exploited in the next section, where a new algebraic structure of information is proposed, which has the algebraic structure derived above.

5.2 Domain-Free Axiomatics

Motivated by the previous section, we introduce here a new algebraic structure, modelling information from a different point of view. Again consider a q-separoid $(D; \leq, \perp)$, whose elements represent domains or questions. Let now Ψ be a set of elements representing pieces of information. This time, however, elements ϕ, ψ, \dots from Ψ are not attached to a specified domain,

they are *domain-free*. Nevertheless, it is assumed to be possible to extract information relative to a domain x in D from every piece of information $\psi \in \Psi$. This will be accomplished by extraction operators $\epsilon_x : \Psi \rightarrow \Psi$ attached to each domain x in D . We assume throughout that $x \neq y$ implies $\epsilon_x \neq \epsilon_y$.

More formally, we assume two operations in Ψ :

1. *Combination*: $\cdot : \Psi \times \Psi \rightarrow \Psi, (\phi, \psi) \mapsto \phi \cdot \psi$,
2. *Extraction*: $\epsilon : \Psi \times D \rightarrow \Psi, (\psi, x) \mapsto \epsilon_x(\psi)$.

As before, combination represents aggregation of information and, new, extraction describes filtering out information relative to a domain. Thus, $\epsilon_x(\psi)$ is thought to be the part of ψ referring to domain or question $x \in D$.

We consider a signature $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ and impose the following axioms:

- A0** *Quasi-Separoid*: $(D; \leq, \perp)$ is a quasi-separoid.
- A1** *Semigroup*: $(\Psi; \cdot)$ is a commutative semigroup with unit 1 and null element 0.
- A2** *Support*: For all $\psi \in \Psi$, there is a domain $x \in D$ such that $\epsilon_x(\psi) = \psi$, and whenever $\epsilon_x(\psi) = \psi$, $x \leq y$, then $\epsilon_y(\psi) = \psi$.
- A3** *Unit and Null*: For all $x \in D$, $\epsilon_x(1) = 1$ and $\epsilon_x(\phi) = 0$ if and only if $\phi = 0$.
- A4** *Extraction*: If $x \perp y | z$ and $\epsilon_x(\psi) = \psi$, then $\epsilon_y(\psi) = \epsilon_y(\epsilon_z(\psi))$
- A5** *Combination*: If $x \perp y | z$ and $\epsilon_x(\phi) = \phi$ and $\epsilon_y(\psi) = \psi$, then $\epsilon_z(\phi \cdot \psi) = \epsilon_z(\phi) \cdot \epsilon_z(\psi)$.

A system $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ satisfying these axioms is called a *domain-free generalised information algebra*. According to the previous section, if $(\Phi, D; \leq, \perp, d, \cdot, t)$ is a labeled information algebra, then $(\phi/\sigma, D; \leq, \perp, \cdot, \epsilon)$ is a domain-free information algebra. So, to any labeled information algebra, a domain-free algebra is associated. Below, in Section 5.3, we shall see that any domain-free information algebra induces a labeled one. This leads to a remarkable duality between the two kinds of algebras. Sometimes, later, we consider also domain-free information algebras without the Support Axiom A2.

There are also domain-free information algebras, which satisfy the following additional axiom:

- A6** *Idempotency*: For all $\phi \in \Phi$ and $x \in D$, $\epsilon_x(\phi) \cdot \phi = \phi$.

Note that idempotency implies, due to the support axiom A2, that $\phi \cdot \phi = \phi$. If this property holds, then the information algebra is called *idempotent*

or *proper*. In fact, in a proper context of information, a repetition of a piece of information or part of it, should give nothing new.

If for a domain x from D , $\epsilon_x(\phi) = \phi$, then x is called a *support* of ϕ . Here follow a few properties of support and extraction:

Lemma 5.2 *If $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is a domain-free generalised information algebra, then the following holds:*

1. x is a support of $\epsilon_x(\phi)$, $\epsilon_x(\epsilon_x(\phi)) = \epsilon_x(\phi)$.
2. If x is a support of both ϕ and ψ , then it is also a support of $\phi \cdot \psi$, $\epsilon_x(\phi \cdot \psi) = \phi \cdot \psi$.
3. If x is a support of ϕ and y of ψ , then $x \vee y$ is a support of $\phi \cdot \psi$, $\epsilon_{x \vee y}(\phi \cdot \psi) = \phi \cdot \psi$.
4. For all $x \in D$, $\epsilon_x(\epsilon_x(\phi) \cdot \psi) = \epsilon_x(\phi) \cdot \epsilon_x(\psi)$.

Proof. 1.) By axiom A2, ϕ has a support y . Since $y \perp x | x$ (C1), axiom A4 implies $\epsilon_x(\epsilon_x(\phi)) = \epsilon_x(\phi)$.

2.) and 3.) From $x \perp x \vee y | x \vee y$ (C1) it follows that $x \perp y | x \vee y$ (C3). If x is a support for ϕ and y for ψ , then axiom A5 implies $\epsilon_{x \vee y}(\phi \cdot \psi) = \epsilon_{x \vee y}(\phi) \cdot \epsilon_{x \vee y}(\psi)$. Further by the support axiom A2, $x \vee y$ is a support of both ϕ and ψ . Thus it follows that $\epsilon_{x \vee y}(\phi \cdot \psi) = \phi \cdot \psi$, which proves item 3 of the lemma. With $x = y$, item 2 follows also.

4.) By axiom A2 any element ψ has a support. So suppose $\epsilon_y(\psi) = \psi$. Then $x \perp y | x$ (C1, C2) and $\epsilon_x(\epsilon_x(\phi)) = \epsilon_x(\phi)$ (item 1 proved above) imply, using axiom A5,

$$\epsilon_x(\epsilon_x(\phi) \cdot \psi) = \epsilon_x(\epsilon_x(\phi)) \cdot \epsilon_x(\psi) = \epsilon_x(\phi) \cdot \epsilon_x(\psi).$$

This concludes the proof. \square

The following is an important and basic example of a domain free information algebra,

Example 5.1 Set Algebras: Let U be any set (of possible worlds) and $(D; \leq)$ a join-subsemilattice of $(\text{Part}(U); \leq)$. By Theorem 2.6 $(D; \leq, \perp)$, where the conditional independence relation is defined as in Section 2.2, is a q-separoid. If the top partition in D is the universe U , then let Ψ be the power set of U , otherwise, let Ψ be the family of subsets S of U which are saturated with respect to a partition P of D , that is, $\sigma_P(S) = S$. As *combination* of elements of Ψ take *intersection*. Clearly, the intersection of a set saturated with respect to partition P_1 and a set saturated with respect to P_2 is a set saturated with respect to $P_1 \vee P_2$. So, Ψ is closed under combination. For extraction take the saturation operators σ_P for $P \in D$. Since $\sigma_P(S)$ is saturated with respect to P , the family Ψ is also closed under

extraction. Axiom A0 is satisfied according to Theorem 2.6, A1 is obvious for intersection, A2 is satisfied by definition of Ψ and since a set saturated for a partition P is also saturated for a finer partition $Q \geq P$. The unit element is the set U and the null element the empty set; both satisfy obviously Axiom A3. Axioms A4 and A5 are proved in (Kohlas & Monney, 1995). This set algebra is clearly the domain-free version of the labeled set algebra in Section 2.3 relative to a f.c.f induced by a join-semilattice of partitions of U . This provides an alternative argument for the validity of Axioms A4 and A5. The domain-free version obtained from the labeled set algebras relative to a general family of compatible frames, obtained by unlabeled as in the previous section, is not a set algebra in the narrow sense given here; its elements are not subsets of some universe U . These domain-free information algebras are *idempotent*. Belief functions provide further examples of domain-free information algebras, albeit no more idempotent ones. \ominus

Just as with labeled information algebras (Section 3.2), an important and interesting special case arises for a domain-free information algebra $(\Psi, D; \leq, \perp_L, \cdot, \epsilon)$, where D is a lattice. In this case the following holds.

Lemma 5.3 *In the domain-free information algebra $(\Psi, D; \leq, \perp_L, \cdot, \epsilon)$, where D is a lattice, for all $x, y \in D$,*

$$\epsilon_y(\epsilon_x(\phi)) = \epsilon_{x \wedge y}(\phi).$$

Proof. We have $x \perp_L y | x \wedge y$. From this and $\epsilon_x(\epsilon_x(\phi)) = \epsilon_x(\phi)$ it follows, using axiom A4,

$$\epsilon_y(\epsilon_x(\phi)) = \epsilon_y(\epsilon_{x \wedge y}(\epsilon_x(\phi))).$$

Assume that z is a support of ϕ . Then $z \perp_L x | x$ (C1) implies $z \perp_L x \wedge y | x$ (C3), and thus again by axiom A4 $\epsilon_{x \wedge y}(\epsilon_x(\phi)) = \epsilon_{x \wedge y}(\phi)$, so $x \wedge y$ is a support of $\epsilon_{x \wedge y}(\phi)$, and $x \wedge y \leq y$, hence indeed by axiom A2 $\epsilon_y(\epsilon_x(\phi)) = \epsilon_{x \wedge y}(\phi)$. \square

This result implies that the extraction operators ϵ_x for $x \in D$ in the algebra $(\Psi, D; \leq, \perp_L, \cdot, \epsilon)$ form a *commutative semigroup* under composition,

$$\epsilon_y(\epsilon_x(\phi)) = \epsilon_x(\epsilon_y(\phi)),$$

which is not the case in general. Moreover, due to the idempotency of extraction these extraction operators form an *idempotent* commutative semigroup, hence a *meet-semilattice*, where we define $\epsilon_x \leq \epsilon_y$ if $\epsilon_y \circ \epsilon_x = \epsilon_x$. Under this partial order it follows indeed that $\epsilon_x \circ \epsilon_y = \epsilon_x \wedge \epsilon_y = \epsilon_{x \wedge y}$ in concordance with Lemma 5.3 and also $\epsilon_x \leq \epsilon_y$ if and only if $x \leq y$. This structure is an instance of an alternative algebraic structure which is defined as follows.

Consider the signature $(\Psi, E; \cdot, \circ)$, where the following operations are defined

1. *Combination*: $\cdot : \Psi \times \Psi \rightarrow \Psi; (\phi, \psi) \mapsto \phi \cdot \psi$.
2. *Extraction*: Any $\epsilon \in E$ is an operator $\epsilon : \Psi \rightarrow \Psi$.

We impose the following axioms on this structure:

- D0** *Semigroup of Extraction*: (E, \circ) is an idempotent, commutative semigroup.
- D1** *Semigroup of Information*: (Ψ, \cdot) is a commutative semigroup with unit 1 and null 0.
- D2** *Support*: For all $\psi \in \Psi$ there is an operator $\epsilon \in E$ such that $\epsilon(\psi) = \psi$.
- D3** *Unit and Null*: For all $\epsilon \in E$, $\epsilon(1) = 1$ and $\epsilon(\phi) = 0$ if and only if $\phi = 0$.
- D4** *Combination*: For all $\epsilon \in E$ and $\phi, \psi \in \Psi$, $\epsilon(\epsilon(\phi) \cdot \psi) = \epsilon(\phi) \cdot \epsilon(\psi)$.

This is related to the domain-free variant of a labeled valuation algebra (see Sections 3.2 and 5.3), and we call it therefore a *domain-free valuation algebra*. Similar, more or less equivalent structures have been introduced by (Shafer, 1991) and are discussed in (Kohlas, 2003a). We may add an additional axiom concerning idempotence:

- D5** *Idempotency*: For all $\psi \in \Psi$ and $\epsilon \in E$, $\epsilon(\psi) \cdot \psi = \psi$.

Such idempotent structures were studied extensively in (Kohlas, 2003a; Kohlas & Schmid, 2014; Kohlas & Schmid, 2016). There are also many examples and generic construction methods for these algebras to be found there. Such idempotent valuation algebras were called information algebras in those references.

As above, axiom D0 implies that E is a meet-semilattice where $\epsilon \leq \eta$ if $\eta \circ \epsilon = \epsilon$. In this order $\epsilon \circ \eta$ is the infimum of ϵ and η , $\epsilon \circ \eta = \epsilon \wedge \eta$. For later reference we add a few properties derived from the axioms of a domain-free valuation algebra.

Lemma 5.4 *Let $(\Psi, E; \cdot, \circ)$ be a domain-free valuation algebra. Then*

1. $\epsilon(\psi) = \psi$ and $\epsilon \leq \eta$ imply $\eta(\psi) = \psi$.
2. $\epsilon \leq \eta$ implies $\epsilon(\psi) = \epsilon(\eta(\psi))$.
3. $\epsilon(\psi) = \eta(\psi) = \psi$ implies $(\epsilon \wedge \eta)(\psi) = \psi$.
4. $\epsilon(\psi) = \psi$ implies $(\epsilon \wedge \eta)(\psi) = \eta(\psi)$.
5. $\epsilon(\phi) = \phi$ and $\epsilon(\psi) = \psi$ imply $\epsilon(\phi \cdot \psi) = \phi \cdot \psi$.

Proof. 1.) We have $\eta(\psi) = \eta(\epsilon(\psi)) = (\epsilon \wedge \eta)(\psi)$. But $\epsilon \wedge \eta = \epsilon$. Therefore, $\eta(\psi) = \epsilon(\psi) = \psi$.

2.) Here we use again that $\epsilon \wedge \eta = \epsilon$ and conclude that $\epsilon(\eta(\psi)) = (\epsilon \wedge \eta)(\psi) = \epsilon(\psi)$.

3.) Since $\epsilon \circ \eta = \epsilon \wedge \eta$ we have $(\epsilon \wedge \eta)(\psi) = \epsilon(\eta(\psi)) = \psi$.

4.) Again, from $\epsilon \circ \eta = \epsilon \wedge \eta$ we obtain $(\epsilon \wedge \eta)(\psi) = \eta(\epsilon(\psi)) = \eta(\psi)$.

5.) Here we use the combination axiom D4 and see that

$$\epsilon(\phi \cdot \psi) = \epsilon(\epsilon(\phi) \cdot \psi) = \epsilon(\phi) \cdot \epsilon(\psi) = \phi \cdot \psi.$$

This concludes the proof. \square

Note that in the framework of a domain-free valuation algebra the property stated in item 1 may be derived, whereas in the case of a generalised domain-free information algebra it must be imposed as an axiom. We mention also, that a set algebra becomes a valuation algebra if the partitions commute in the sub-join-semilattice $(D; \leq)$ of a partition lattice $(part(U), \leq)$ (Kohlas & Schmid, 2016).

In many cases E is not only a semilattice, but a lattice. In particular this is the case for the domain-free algebra $(\Psi; D; \leq, \perp_L, \cdot, t)$ introduced above. Then some more results about extraction can be obtained.

Lemma 5.5 *Let $(\Psi, E; \cdot, \circ)$ be a domain-free valuation algebra, where $(E; \leq)$ is a lattice under the order induced by the idempotent, commutative semigroup (E, \circ) and $\epsilon \perp_L \eta|_\chi$ is the corresponding q -separoid in this lattice. Then*

1. $\epsilon(\psi) = \psi$ and $\epsilon \perp_L \eta|_\chi$ imply $\eta(\psi) = \eta(\chi(\psi))$.
2. $\epsilon(\phi) = \phi$, $\eta(\psi) = \psi$ and $\epsilon \perp_L \eta|_\chi$ imply $\chi(\phi \cdot \psi) = \chi(\phi) \cdot \chi(\psi)$.
3. $\epsilon(\phi) = \phi$ and $\eta(\psi) = \psi$ imply $(\epsilon \vee \eta)(\phi \cdot \psi) = \phi \cdot \psi$.

Proof. 1.) From $\epsilon \leq \epsilon \vee \chi$ and $\eta \leq \eta \vee \chi$, it follows, using items 1 and 2 of Lemma 5.4, and that composition is infimum in the lattice $(E; \leq)$,

$$\eta(\psi) = \eta((\epsilon \vee \chi)(\psi)) = \eta((\eta \vee \chi)((\epsilon \vee \chi)(\psi))) = \eta((\epsilon \vee \chi) \wedge (\eta \vee \chi)(\psi)).$$

But $\epsilon \perp_L \eta|_\chi$ implies $(\epsilon \vee \chi) \wedge (\eta \vee \chi) = \chi$ and therefore, indeed, $\eta(\psi) = \eta(\chi(\psi))$.

2.) and 3.) Again, by Lemma 5.4, we have $\phi = (\epsilon \vee \chi)(\phi)$ and $\psi = (\eta \vee \chi)(\psi)$. Therefore $\chi(\phi \cdot \psi) = \chi((\epsilon \vee \chi)(\phi) \cdot (\eta \vee \chi)(\psi))$. From $\chi \leq \epsilon \vee \chi$ and Axiom D4 we obtain

$$\begin{aligned} \chi(\phi \cdot \psi) &= \chi((\epsilon \vee \chi)((\epsilon \vee \chi)(\phi) \cdot (\eta \vee \chi)(\psi))) \\ &= \chi((\epsilon \vee \chi)(\phi) \cdot ((\epsilon \vee \chi) \wedge (\eta \vee \chi)(\psi))) \end{aligned}$$

Applying $\epsilon \perp_L \eta|_\chi$, Axiom D4 and $\phi = (\epsilon \vee \chi)(\phi)$ gives then

$$\chi(\phi \cdot \psi) = \chi(\phi \cdot \chi(\psi)) = \chi(\phi) \cdot \chi(\psi).$$

This proves item 2. Item 3 follows from item 2 since $\epsilon \perp_L \eta | \epsilon \vee \eta$ and $\phi = (\epsilon \vee \eta)(\phi)$, $\psi = (\epsilon \vee \eta)(\psi)$. \square

These lemmas show that a domain-free valuation algebra $(\Psi, E; \cdot, \circ)$ induces a domain-free information algebra $(\Psi, E; \leq, \perp_L, \cdot, \epsilon)$ if E is a lattice. The situation in the domain-free view is therefore rather the same as in the labeled view. This aspect will be clarified in the next section. Note however that if E is only a semilattice in the domain-free valuation algebra $(\Psi, E; \circ, \cdot)$, then it is not a generalised information algebra. So, it should be kept in mind that generalised information algebras and valuation algebras are different concepts in general, although identical in some particular cases.

5.3 Duality

We have seen above that a domain-free information algebra can be derived from a labeled one. It turns out that, conversely, a labeled information algebra may also be obtained from a domain-free one. This establishes then a *duality* between labeled and domain-free information algebras. This duality applies also to the special case of valuation algebras, although with the reservation that the extraction operators form a lattice, not only a semilattice.

Assume $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ to be a domain-free generalised information algebra. Define

$$\Phi_x = \{(\phi, x) : \epsilon_x(\phi) = \phi\}, \quad \Phi = \bigcup_{x \in D} \Phi_x.$$

We define the following operations relative to the signature $(\Phi, D; \leq, \perp, d, \cdot, t)$:

1. *Labeling*: $d : \Phi \rightarrow D; (\phi, x) \mapsto d(\phi, x) = x$.
2. *Combination*: $\cdot : \Phi \times \Phi \rightarrow \Phi; ((\phi, x), (\psi, y)) \mapsto (\phi, x) \cdot (\psi, y) = (\phi \cdot \psi, x \vee y)$.
3. *Transport*: $t : \Phi \times D \rightarrow \Phi; ((\phi, x), y) \mapsto t_y(\phi, x) = (\epsilon_y(\phi), y)$.

Note that we use the same symbol \cdot for combination in Ψ and in Φ ; it will always be clear from the context which operation is meant. We remark that due to the results of the previous section all these operations are well defined

We claim that $(\Phi, D; \leq, \perp, \cdot, t)$ is a labeled generalised information algebra. Here is the verification of the axioms: The structure $(D; \leq, \perp)$ is a q-separoid by definition (A0). Combination in (Φ, \cdot) is clearly associative and commutative, so (Φ, \cdot) is a commutative semigroup (A1). By definition of Combination, Labeling and Transport in Φ we have $d((\phi, x) \cdot (\psi, y)) = d(\phi \cdot \psi, x \vee y) = x \vee y = d(\phi, x) \vee d(\psi, y)$ and $d(t_y(\phi, x)) = d(\epsilon_y(\phi), y) = y$. So the Labeling Axiom (A2) is satisfied. The null and unit elements associated with $x \in D$ are $(0, x)$ and $(1, x)$. By the definition of Combination

and Transport in Φ , we obtain $t_x(\phi, y) = (\epsilon_x(\phi), x) = (0, x)$ if and only if $\phi = 0$, $(\phi, y) \cdot (1, x) = (\phi, x \vee y) = (\epsilon_{x \vee y}(\phi), x \vee y) = t_{x \vee y}(\phi, y)$ and $(1, x) \cdot (1, y) = (1, x \vee y)$. This confirms the Unit and Null Axiom (A3).

For the Transport Axiom (A4) assume $x \perp y | z$ and consider an element (ϕ, x) such that $d(\phi, x) = x$ and $\epsilon_x(\phi) = \phi$. Then by the Extraction Axiom (A4) of the domain-free algebra

$$t_y(\phi, x) = (\epsilon_y(\phi), y) = (\epsilon_y(\epsilon_z(\phi)), y) = t_y(t_z(\phi, x)).$$

This is the Transport Axiom (A4) in the labeled version.

To verify the Combination Axiom (A5) assume $x \perp y | z$ and consider elements (ϕ, x) and (ψ, y) such that $d(\phi, x) = x$ and $d(\psi, y) = y$. Then, by the definitions of Combination and Transport in Φ , and invoking Combination Axiom (A5) for the domain-free algebra,

$$\begin{aligned} t_z((\phi, x) \cdot (\psi, y)) &= (\epsilon_z(\phi \cdot \psi), z) = (\epsilon_z(\phi) \cdot \epsilon_z(\psi), z) \\ &= (\epsilon_z(\phi), z) \cdot (\epsilon_z(\psi), z) = t_z(\phi, x) \cdot t_z(\psi, y). \end{aligned}$$

This is the Combination Axiom (A4) in the labeled version.

The Identity Axiom (A6) follows from $t_x(\phi, x) = (\epsilon_x(\phi), x) = (\phi, x)$ since by definition x is a support of ϕ . So, the algebra $(\Phi, D; \leq, \perp, d, \cdot, t)$ is indeed an instance of a labeled information algebra. If $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is idempotent, then so is $(\Phi, D; \leq, \perp, d, \cdot, t)$. Consider (ϕ, x) and $y \leq x$, then $(\phi, x) \cdot t_y(\phi, x) = (\phi, x) \cdot (\epsilon_y(\phi), y) = (\phi \cdot \epsilon_y(\phi), x \vee y) = (\phi, x)$. This is the Idempotency axiom A7 of the labeled information algebra.

In the example of a domain-free set algebra associated with a universe U and a q-separoid $(D; \leq, \perp)$ of partitions, the elements of the corresponding labeled algebra are the pairs $(S; P)$ where S is a subset of U , saturated relative to the partition P . We might as well replace S by the set S' of the blocks of P it is composed of and replace partition P by the frame Θ_P . So, this gives us the labeled algebra of pairs (S', Θ_P) , where S' is a subset of the frame Θ_P . This is essentially the labeled set algebra relative to the f.c.f of the frames Θ_P for $P \in D$.

We may now start with a labeled information algebra \mathbf{L} , say $\mathbf{L} = (\Phi, D; \leq, \perp, d, \cdot, t)$ and derive its domain-free version $\mathbf{DL} = (\Phi/\sigma, D; \leq, \perp, \cdot, \epsilon)$ in the way described in Section 5.1. In a further step we may construct from \mathbf{DL} its labeled version $\mathbf{LDL} = (\Phi', D; \leq, \perp, d', \cdot, t')$ as shown above. In the same way we may start with a domain-free information algebra $\mathbf{D} = (\Psi, D; \leq, \perp, \cdot, \epsilon)$ and get the associated labeled algebra $\mathbf{LD} = (\Phi, D; \leq, \perp, d, \cdot, t)$ and then obtain from this labeled algebra its domain-free version $\mathbf{DLD} = (\Phi/\sigma, D; \leq, \perp, \cdot, \epsilon')$. It may be suspected that \mathbf{L} and \mathbf{LDL} are essentially the same algebra, and so are \mathbf{D} and \mathbf{DLD} . In order to make this statement precise we need to define the notion of isomorphisms between labeled information algebra on the one hand and between domain-free information algebras on the other hand.

Consider two labeled information algebras $\mathbf{L} = (\Phi, D; \leq, \perp, d, \cdot, t)$ and $\mathbf{L}' = (\Phi', D; \leq, \perp, d', \cdot', t')$. To simplify, we assume that both algebras are based on the same q-separoid; this corresponds to the situation we are interested in. Let $T = \{t_x : x \in D\}$ and $T' = \{t'_x : x \in D\}$ be the sets of the transport operators in the two algebras. We consider a pair of maps

$$f : \Phi \rightarrow \Phi', \quad g : T \mapsto T' \text{ such that } g(t_x) = t'_x \text{ for all } x \in D.$$

If the following conditions are satisfied, the pair of maps (f, g) is called a *labeled information algebra homomorphism*:

1. $f(\phi \cdot \psi) = f(\phi) \cdot' f(\psi)$, for all $\phi, \psi \in \Phi$,
2. $f(0_x) = 0'_x$ and $f(1_x) = 1'_x$, where $0'_x$ and $1'_x$ are the null elements and unities in (Φ', \cdot') ,
3. $f(t_x(\phi)) = g(t_x)(f(\phi))$.

Note that the map g is by definition one-to-one and onto. If the map f is also onto and one-to-one, the pair (f, g) is called a *labeled information algebra isomorphism* and the two algebras are called *isomorphic*, written as $\mathbf{L} \cong \mathbf{L}'$.

Similarly, for domain-free information algebras: Let $\mathbf{D} = (\Psi, D; \leq, \perp, \cdot, \epsilon)$ and $\mathbf{D}' = (\Psi', D; \leq, \perp, \cdot', \epsilon')$. Let $E = \{\epsilon_x : x \in D\}$ and $E' = \{\epsilon'_x : x \in D\}$ be the sets of the extractor operators in the two algebras. Again a pair (f, g) of maps

$$f : \Psi \rightarrow \Psi', \quad g : E \mapsto E' \text{ such that } g(\epsilon_x) = \epsilon'_x \text{ for all } x \in D.$$

satisfying

1. $f(\phi \cdot \psi) = f(\phi) \cdot' f(\psi)$, for all $\phi, \psi \in \Psi$,
2. $f(0) = 0'$ and $f(1) = 1'$, where $0'$ and $1'$ are the null and unity in (Ψ', \cdot') ,
3. $f(\epsilon_x(\phi)) = g(\epsilon_x)(f(\phi))$.

is called a *domain-free information algebra homomorphism*. The map g is still one-to-one and onto. If the map f is onto and one-to-one, then (f, g) is called a *domain-free information algebra isomorphism*; \mathbf{D} and \mathbf{D}' are called *isomorphic*, written $\mathbf{D} \cong \mathbf{D}'$.

We are now going to show that \mathbf{L} and \mathbf{LDL} are isomorphic and so are \mathbf{D} and \mathbf{DLD} . In the first case, consider the maps (f, g) from \mathbf{L} into \mathbf{LDL} , defined by

$$\begin{aligned} f(\phi) &= ([\phi]_\sigma, x), \text{ if } d(\phi) = x, \\ g(t_x) &= t'_x, \text{ where } t'_x([\phi]_\sigma, y) = (\epsilon_x([\phi]_\sigma), x), \end{aligned} \quad (5.3)$$

Similarly, we define the pair (f, g) of maps from \mathbf{D} into $\mathbf{DL}\mathbf{D}$,

$$\begin{aligned} f(\psi) &= [(\psi, x)]_\sigma, \text{ if } \epsilon_x(\psi) = \psi, \\ g(\epsilon_x) &= \epsilon'_x, \text{ where } \epsilon'_x([(\psi, y)]_\sigma) = [t_x(\psi, y)]_\sigma \end{aligned} \quad (5.4)$$

We claim that these pairs of maps are respectively labeled information algebra and domain-free information algebra isomorphisms.

Theorem 5.3 *We have $\mathbf{L} \cong \mathbf{LDL}$ and $\mathbf{D} \cong \mathbf{DL}\mathbf{D}$ and the respective pairs of maps (f, g) (5.3) and (5.4) are the isomorphisms.*

Proof. First, consider the labeled case, the pair of maps defined in (5.3). Here we have first, assuming $d(\phi) = x$ and $d(\psi) = y$,

$$\begin{aligned} f(\phi \cdot \psi) &= ([\phi \cdot \psi]_\sigma, x \vee y) = ([\phi]_\sigma \cdot [\psi]_\sigma, x \vee y) \\ &= ([\phi]_\sigma, x) \cdot' ([\psi]_\sigma, y) = f(\phi) \cdot' f(\psi). \end{aligned}$$

Since $f(0_x) = ([0_x]_\sigma, x)$ and $f(1_x) = ([1_x]_\sigma, x)$, null and unit elements are preserved. Further assume $d(\phi) = y$. Then

$$f(t_x(\phi)) = ([t_x(\phi)]_\sigma, x) = (\epsilon_x([\phi]_\sigma), x) = t'_x([\phi]_\sigma, y) = g(t_x)(f(\phi)).$$

This proves that the pair of maps (f, g) is a labeled information algebra homomorphism. Consider any element $([\phi]_\sigma, x)$ in \mathbf{LDL} . Then, this is the image of the element ϕ from \mathbf{L} , so the map f is onto. Further, $([\phi]_\sigma, x) = ([\psi]_\sigma, y)$ implies $x = y$ and $[\phi]_\sigma = [\psi]_\sigma$. By definition of the map f , $d(\phi) = x$ and $d(\psi) = y = x$. But this, together with $[\phi]_\sigma = [\psi]_\sigma$, implies $\phi = \psi$. The map f is therefore one-to-one. Thus, the pair (f, g) is a labeled information algebra isomorphism and therefore $\mathbf{L} \cong \mathbf{LDL}$.

Second, consider the domain-free case, the pair of maps defined in (5.4). Let ϕ and ψ be elements from \mathbf{D} with supports x and y respectively. Then

$$\begin{aligned} f(\phi \cdot \psi) &= [(\phi \cdot \psi, x \vee y)]_\sigma = [(\phi, x) \cdot (\psi, y)]_\sigma \\ &= [(\phi, x)]_\sigma \cdot' [(\psi, y)]_\sigma = f(\phi) \cdot' f(\psi). \end{aligned} \quad (5.5)$$

Further, $f(0) = [(0, x)]_\sigma$ and $f(1) = [(1, x)]_\sigma$ are clearly the null and unit elements in $\mathbf{DL}\mathbf{D}$. Next, assume that y is a support of the element ψ in \mathbf{D} . Then

$$f(\epsilon_x(\psi)) = [(\epsilon_x(\psi), x)]_\sigma = [t_x(\psi, y)]_\sigma = \epsilon'_x([(\psi, y)]_\sigma) = g(\epsilon_x)(f(\psi)).$$

So the pair (f, g) is a domain-free information algebra homomorphism. If $[(\psi, x)]_\sigma$ is an element from $\mathbf{DL}\mathbf{D}$, then x is a support of ψ and f maps ψ to $[(\psi, x)]_\sigma$. So, the map f is onto. Assume that $[(\phi, x)]_\sigma = [(\psi, y)]_\sigma$. Then x and y are supports of ϕ and ψ respectively; and $(\phi, x) \equiv_\sigma (\psi, y)$ means that $(\phi, x \vee y) = (\psi, x \vee y)$. This shows that $\phi = \psi$; the map f is one-to-one.

Therefore, the pair (f, g) is a domain-free information algebra isomorphism, and $\mathbf{D} \cong \mathbf{DLD}$. \square

According to this theorem, labeled and domain-free algebras are *dual* in the technical sense of the theorem above. We may freely pass from labeled to domain-free algebras and back. These two kinds of algebras are the two sides of the same coin. Labeled information algebras are better suited for computational purposes, whereas domain-free information algebras usually are preferred for theoretical studies.

For labeled valuation algebras and domain-free valuation algebras a similar duality, the special case of the duality introduced above has been shown in (Kohlas, 2003a), provided that the semigroup of extractor operators E in the domain-free case is a lattice and provided the labeled valuation algebras are stable. So, for example probability potentials, densities and Gaussian densities have no domain-free version.

Chapter 6

Order of Information

6.1 The Idempotent Case

Information may be, in informal terms, more or less precise, more or less informative. This should be reflected by some order between pieces of information. Such orders are the subject of the present section. Information order can be studied both in labeled or domain-free information algebras. We propose to base our discussion on domain-free information algebras.

Let then $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be a domain-free generalised information algebra. The basic idea is that a piece of information is more informative than an other one, if one needs to add a further piece of information to the second one to get the first one. So, we define, for $\phi, \psi \in \Psi$,

$$\phi \leq \psi, \text{ iff there exists } \chi \in \Psi \text{ such that } \psi = \phi \cdot \chi. \quad (6.1)$$

This relation satisfies

1. *Reflexivity*: $\psi \leq \psi$, since $\psi = \psi \cdot 1$,
2. *Transitivity*: $\phi \leq \psi$ and $\psi \leq \eta$ imply $\phi \leq \eta$, since $\psi = \phi \cdot \chi_1$, $\eta = \psi \cdot \chi_2$ imply $\eta = \phi \cdot \chi_1 \cdot \chi_2$.

Antisymmetry however does not hold in general. Therefore, the relation \leq defined in (6.1) is a *preorder* in Ψ .

If the information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is *idempotent*, then $\phi \leq \psi$ if and only if $\phi \cdot \psi = \psi$. In fact, $\psi = \phi \cdot \chi$, gives by idempotency, if both sides are combined by ψ , $\psi = (\phi \cdot \chi) \cdot \psi = \phi \cdot (\phi \cdot \chi) \cdot \psi = \phi \cdot \psi \cdot \psi = \phi \cdot \psi$. In idempotent information algebras, the relation \leq is a *partial order*, since $\phi \leq \psi$ and $\psi \leq \phi$ imply $\phi = \psi \cdot \phi = \psi$. Here $\phi \leq \psi$ means that nothing is gained if the piece of information ϕ is added to ψ , the information in ϕ is already covered by ψ . Note that in this idempotent case

1. $1 \leq \psi \leq 0$ for all $\psi \in \Psi$,

2. $\phi, \psi \leq \phi \cdot \psi$,
3. $\phi \leq \psi$ implies $\phi \cdot \eta \leq \psi \cdot \eta$ for all $\eta \in \Psi$,
4. $\epsilon_x(\psi) \leq \psi$ for all $x \in D$ and $\psi \in \Psi$,
5. $\phi \leq \psi$ implies $\epsilon_x(\phi) \leq \epsilon_x(\psi)$ for all $x \in D$,
6. $x \leq y$ implies $\epsilon_x(\psi) \leq \epsilon_y(\psi)$ for all $\psi \in \Psi$.

These are clearly properties one would expect from an information order in general: Vacuous information is least informative, contradiction (which properly speaking is not an information) is the greatest element in the information order; combined information is more informative than each of its parts, the order is compatible with combination and extraction of information does not increase information.

Note that the preorder defined in (6.1), satisfies the first three of these requirements. The other ones are not guaranteed in general and need special consideration. This will be discussed in the following sections. The partial order in idempotent information algebra really is a neat order of information. Therefore we call idempotent information algebra also *proper information algebras*. This order has been discussed and studied in detail for the case of idempotent valuation algebras in (Kohlas, 2003a; Kohlas & Schmid, 2014; Kohlas & Schmid, 2016). Many of the results for this case carry over to generalised information algebras. This will be discussed in Section 7. In the following two sections, we present two interesting cases, where the preorder satisfies also the other requirement stated above. This will also show the relation of the preorder to the partial order of idempotent information and illuminate the limits of the preorder.

6.2 Regular Algebras

Order in semigroup theory has been studied in several papers, we cite only two of them, (Nambooripad, 1980; Mitsch, 1986). These papers study natural order, that is an order, which can be defined in terms of the operations of the semigroup. This is exactly the case of our definition above. Of particular interest in these theories are *regular* semigroups. In the context of valuation algebras, such regular semigroups or rather the generalisation of them to valuation algebras, turned out to be of interest in two respects: They allow to introduce partial division into the algebra, which allows to adapt local computation architectures known for Bayesian networks to valuation algebras (Lauritzen & Jensen, 1997; Kohlas, 2003a). Secondly, this division permits also to generalise conditioning, as known in probability, to valuation algebras (Kohlas, 2003a). Now, as we shall see in this and the subsequent section, this is relevant for information order too.

We summarise here the theory of regular semigroups and adapt it to *regular generalised information algebras*, generalising the theory of regular valuation algebras (Kohlas, 2003a) and discuss how this applies to information order as defined in (6.1). We start with the definition of regularity in information algebras.

Definition 6.1 *Regular Information Algebras:* Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be a generalised domain-free information algebra. An element $\psi \in \Psi$ is called *regular*, if for all $x \in D$ there is an element $\chi \in \Psi$ with support x such that

$$\psi = \epsilon_x(\psi) \cdot \chi \cdot \psi. \quad (6.2)$$

The information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is called *regular*, if all its elements are regular.

Of course, the element χ above in the definition of regularity depends both on x and ψ , although we do not express this dependence explicitly. If y is a support of ψ , then regularity implies also

$$\psi = \psi \cdot \chi \cdot \psi. \quad (6.3)$$

This is the definition of regularity in a semigroup $(\Psi; \cdot)$ and establishes the link to semigroup theory, see for example (Clifford & Preston, 1967) and the work cited above. Note that in these references semigroups are not assumed to be commutative, as is the case here.

In this section we assume that $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is regular. Two elements ϕ and ψ from Ψ are called *inverses*, if

$$\phi = \phi \cdot \psi \cdot \phi \text{ and } \psi = \psi \cdot \phi \cdot \psi \quad (6.4)$$

We keep with the notation in the literature, although in our commutative case we could also have written $\phi = \phi \cdot \phi \cdot \psi, \dots$. Note that $\phi \leq \psi$ and $\psi \leq \phi$ if ϕ and ψ are inverses.

The following results are well-known from semigroup theory (see for instance (Kohlas, 2003a)): If $\phi = \phi \cdot \psi \cdot \phi$, then ϕ and $\psi \cdot \phi \cdot \psi$ are inverses. Each element of a regular semigroup has thus an inverse, and this inverse is unique. If ϕ and ψ are inverses, then $f = \phi \cdot \psi$ is an idempotent element, $f \cdot f = f$. If S is a subset of Ψ , define $\psi \cdot S$ to be the set $\{\psi \cdot \phi : \phi \in S\}$. The set $\psi \cdot \Psi$ is called the *principal filter* generated by ψ , since $\psi \cdot \Psi = \{\psi \cdot \chi : \chi \in \Psi\} = \{\phi : \psi \leq \phi\}$. There exists for any $\psi \in \Psi$ a unique idempotent f_ψ such that $\psi \cdot \Psi = f_\psi \cdot \Psi$. The *Green relation* is defined as

$$\phi \equiv_\gamma \psi \text{ if } \phi \cdot \Psi = \psi \cdot \Psi. \quad (6.5)$$

It is an equivalence relation in Ψ . Its equivalence classes $[\psi]_\gamma$ are *groups* for all $\psi \in \Psi$ (Kohlas, 2003a). So Ψ is a union of disjoint groups. The

unit element of the group $[\phi]_\gamma$ is the idempotent f_ϕ and for any $\psi \in [\phi]_\gamma$ its inverse in the semigroup is the inverse in $[\phi]_\gamma$. Note that if ϕ and ψ are Green-equivalent, then $\phi \leq \psi$ and $\psi \leq \phi$.

Consider now the idempotents $F = \{f_\psi : \psi \in \Psi\}$. They form an idempotent sub-semigroup of $(\Psi; \cdot)$. According to Section 6.1 they are partially ordered by $f_\phi \leq f_\psi$ if $f_\phi \cdot f_\psi = f_\psi$ ¹. The unit 1 and the null element 0 are idempotents, $0, 1 \in F$. So, the idempotents F form a bounded semilattice where $f_\phi \cdot f_\psi = f_\phi \vee f_\psi$. Further, we have also

$$f_\phi \cdot f_\psi = f_{\phi \cdot \psi}. \quad (6.6)$$

Since the idempotents f_ϕ uniquely represent their class $[\phi]_\gamma$, we may also define a partial order among classes by $[\phi]_\gamma \leq [\psi]_\gamma$ if $f_\phi \leq f_\psi$. Then we obtain

$$[\phi \cdot \psi]_\gamma = [\phi]_\gamma \vee [\psi]_\gamma. \quad (6.7)$$

We summarise now some results about preorder in Ψ and partial order among idempotents in F and among the classes $[\phi]_\gamma$.

Lemma 6.1 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be a regular generalised information algebra. Then*

1. $\phi \leq \psi$ iff $[\phi]_\gamma \leq [\psi]_\gamma$,
2. $\phi \leq \psi$ iff $\psi \cdot \Psi = \phi \cdot \psi \cdot \Psi$,
3. $\phi \leq \psi$ iff $\psi \cdot \Psi \subseteq \phi \cdot \Psi$,
4. $\phi \leq \psi$ and $\psi \leq \phi$ iff $\phi \equiv_\gamma \psi$,

Proof. 1.) Assume $\phi \leq \psi$, that is $\phi \cdot \chi = \psi$. Then $[\phi \cdot \chi]_\gamma = [\phi]_\gamma \vee [\chi]_\gamma = [\psi]_\gamma$. This shows that $[\phi]_\gamma \leq [\psi]_\gamma$.

Conversely, assume $[\phi]_\gamma \leq [\psi]_\gamma$ such that $[\phi \cdot \psi]_\gamma = [\phi]_\gamma \vee [\psi]_\gamma = [\psi]_\gamma$. This means that $\psi \cdot \Psi = \phi \cdot \psi \cdot \Psi$, hence $\psi \in \phi \cdot \psi \cdot \Psi$, therefore $\psi = \phi \cdot \psi \cdot \chi$ for some χ . But this means that $\phi \leq \psi$.

2.) We have just proved that $\psi \cdot \Psi = \phi \cdot \psi \cdot \Psi$ implies $\phi \leq \psi$. Assume then that $\phi \leq \psi$. By item 1 we have also $f_\phi \leq f_\psi$ or $f_\phi \cdot f_\psi = f_{\phi \cdot \psi} = f_\psi$. But then $\psi \cdot \Psi = f_\psi \cdot \Psi = f_{\phi \cdot \psi} \cdot \Psi = \phi \cdot \psi \cdot \Psi$.

3.) If $\phi \leq \psi$, then $\psi = \phi \cdot \chi$. Consider $\eta \in \psi \cdot \Psi$, then $\eta = \psi \cdot \chi' = \phi \cdot \chi \cdot \chi'$. So $\eta \in \phi \cdot \Psi$. Conversely, if $\psi \cdot \Psi \subseteq \phi \cdot \Psi$, then $\psi \in \phi \cdot \Psi$, hence there is a χ such that $\psi = \phi \cdot \chi$, and thus $\phi \leq \psi$.

4.) We have by item 2 $\phi \leq \psi$ iff $\psi \cdot \Psi = \phi \cdot \psi \cdot \Psi$ and $\psi \leq \phi$ iff $\phi \cdot \Psi = \phi \cdot \psi \cdot \Psi$. Therefore, $\phi \cdot \Psi = \psi \cdot \Psi$, hence $\phi \equiv_\gamma \psi$. \square

So far, this is essentially semigroup theory. We now consider extraction and extend thus this order theory to information algebras. Here is a first important result:

¹This order is the opposite to the one usually considered in the literature, but it corresponds better to our purposes of information order, as we shall see.

Theorem 6.1 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be a regular generalised information algebra. The Green relation \equiv_γ is a congruence relative to combination and extraction in the algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$*

Proof. The relation \equiv_γ is an equivalence relation. If $\phi \equiv_\gamma \psi$, then $[\phi]_\gamma = [\psi]_\gamma$. Consider any element η of Ψ . Then $[\phi]_\gamma \vee [\eta]_\gamma = [\psi]_\gamma \vee [\eta]_\gamma$, hence $[\phi \cdot \eta]_\gamma = [\psi \cdot \eta]_\gamma$ and thus $\phi \cdot \eta \equiv_\gamma \psi \cdot \eta$.

Assume again $\phi \equiv_\gamma \psi$ such that $\phi \cdot \Psi = \psi \cdot \Psi$, and consider the operator ϵ_x . From $\phi \in \phi \cdot \Psi$ we conclude that $\phi = \psi \cdot \chi$ for some $\chi \in \Psi$ and therefore $\epsilon_x(\phi) = \epsilon_x(\psi \cdot \chi)$. By regularity we have $\psi = \epsilon_x(\psi) \cdot \chi' \cdot \psi$ and thus $\epsilon_x(\phi) = \epsilon_x(\epsilon_x(\psi) \cdot \chi \cdot \chi' \cdot \psi) = \epsilon_x(\psi) \cdot \epsilon_x(\chi \cdot \chi' \cdot \psi)$. This shows that $\epsilon_x(\psi) \leq \epsilon_x(\phi)$. By symmetry we have also $\epsilon_x(\phi) \leq \epsilon_x(\psi)$, therefore, by Lemma 6.1 item 4, $\epsilon_x(\phi) \equiv_\gamma \epsilon_x(\psi)$. This proves that \equiv_γ is a congruence. \square

Here follow a few results on order and extraction, which show some desirable results, in particular the validity of the expected properties 4.) to 6.) of an information order formulated above (Section 6.1).

Theorem 6.2 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be a regular generalised information algebra. Then*

1. $\epsilon_x(\psi) \leq \psi$ for all $x \in D$ and $\psi \in \Psi$.
2. $\phi \leq \psi$ implies $\epsilon_x(\phi) \leq \epsilon_x(\psi)$ for all $x \in D$.
3. $x \leq y$ implies $\epsilon_x(\psi) \leq \epsilon_y(\psi)$ for all $\psi \in \Psi$.

Proof. 1.) By regularity $\psi = \psi \cdot \chi \cdot \epsilon_x(\psi)$ where $\epsilon_x(\chi) = \chi$. Applying the extraction operator on both sides gives $\epsilon_x(\psi) = \epsilon_x(\psi) \cdot \epsilon_x(\psi) \cdot \chi$, hence $\epsilon_x(\psi) \geq \chi$ and therefore $[\epsilon_x(\psi)]_\gamma \geq [\chi]_\gamma$ (Lemma 6.1). From the regularity formula we obtain also $[\psi]_\gamma = [\psi]_\gamma \vee [\chi]_\gamma \vee [\epsilon_x(\psi)]_\gamma = [\psi]_\gamma \vee [\epsilon_x(\psi)]_\gamma$, hence $[\epsilon_x(\psi)]_\gamma \leq [\psi]_\gamma$. This implies $\epsilon_x(\psi) \leq \psi$ (Lemma 6.1).

2.) If $\phi \leq \psi$, then $\psi \cdot \Psi = \phi \cdot \psi \cdot \Psi$ (Lemma 6.1). This implies $\psi = \psi \cdot \phi \cdot \chi$ for some $\chi \in \Psi$. By regularity we have $\phi = \phi \cdot \epsilon_x(\phi) \cdot \mu$ and $\psi = \psi \cdot \epsilon_x(\psi) \cdot \mu'$, where x is a support of both μ and μ' . From this we deduce

$$\begin{aligned} \epsilon_x(\psi) &= \epsilon_x(\psi \cdot \phi \cdot \chi) \\ &= \epsilon_x(\epsilon_x(\psi) \cdot \epsilon_x(\phi) \cdot \mu \cdot \mu' \cdot \psi \cdot \phi \cdot \chi) \\ &= \epsilon_x(\psi) \cdot \epsilon_x(\phi) \cdot \epsilon_x(\mu \cdot \mu' \cdot \psi \cdot \phi \cdot \chi) \end{aligned} \quad (6.8)$$

This proves that $\epsilon_x(\phi) \leq \epsilon_x(\psi)$.

3.) Assume that z is a support of ψ . By C1, $z \perp y | y$ and by C3 $z \perp x | y$, since $x \leq y$. It follows from Axiom A4 that $\epsilon_x(\psi) = \epsilon_x(\epsilon_y(\psi))$. Then item 1 above shows that $\epsilon_x(\psi) \leq \epsilon_y(\psi)$. \square

Based on Theorem 6.1, we may consider the quotient algebra $(\Psi/\gamma, D; \leq, \cdot, \epsilon)$, which by general results of universal algebra must still be a generalised

information algebra. In fact, we define the following operations between classes

1. *Combination*: $[\phi]_\gamma \cdot [\psi]_\gamma = [\phi \cdot \psi]_\gamma$,
2. *Extraction*: $\epsilon_x([\psi]_\gamma) = [\epsilon_x(\psi)]_\gamma$.

We denote the operations of combination and extraction in Ψ/γ by the same symbols as in Ψ ; there is no risk of confusion. The projection pair of maps (f, g) , where $f(\psi) = [\psi]_\gamma$ and $g(\epsilon_x) = \epsilon_x$ (meaning at the right hand side, the operator in Ψ/γ) is clearly a homomorphism. A homomorphism maintains order. In addition, it turns out that the information algebra $(\Psi/\gamma, D; \leq, \perp, \cdot, \epsilon)$ is idempotent.

Theorem 6.3 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be a regular generalised information algebra and \equiv_γ the Green relation. Then the quotient algebra $(\Psi/\gamma, D; \leq, \perp, \cdot, \epsilon)$ is an idempotent generalised information algebra, homomorphic to $(\Psi, D; \leq, \perp, \cdot, \epsilon)$.*

Proof. That $(\Psi/\gamma, D; \leq, \cdot, \epsilon)$ is a generalised information follows since the pair of maps defined above form a homomorphism. Idempotency follows from $\epsilon_x[\psi]_\gamma = [\epsilon_x(\psi)]_\gamma \leq [\psi]_\gamma$ (Theorem 6.2), hence $[\psi]_\gamma \cdot \epsilon_x([\psi]_\gamma) = [\psi]_\gamma \vee \epsilon_x([\psi]_\gamma) = [\psi]_\gamma$. \square

Instead of the quotient algebra $(\Psi/\gamma, D; \leq, \perp, \cdot, \epsilon)$ we can also consider the idempotents in the equivalence classes, because there is a one-to-one association between idempotents and their classes. In the signature $(F, D; \leq, \perp, \cdot, \bar{\epsilon})$, where $F = \{f_\psi : \psi \in \Psi\}$, again the two operations of combination and extraction are defined:

1. *Combination*: $f_\phi \cdot f_\psi = f_{\phi \cdot \psi}$,
2. *Extraction*: $\bar{\epsilon}_x(f_\psi) = f_{\epsilon_x(\psi)}$.

This algebra is still an idempotent generalised information algebra, homomorphic to $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. Because of the idempotency, it can be considered as the *deterministic part* of $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ (although it is not a sub-algebra of $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ and $\bar{\epsilon}$ and ϵ are different). By the pair of maps $[\psi]_\gamma \mapsto f_\psi$ and $\epsilon \mapsto \bar{\epsilon}$, the algebras $(\Psi/\gamma, D; \leq, \perp, \cdot, \epsilon)$ and $(F, D; \leq, \perp, \cdot, \bar{\epsilon})$ are isomorphic. We refer to the example of probability potentials below for an illustration.

To conclude this section, we remark that the relation $\psi = \phi \cdot \chi$ if $\phi \leq \psi$, in the context of labeled algebras, can be linked to conditionals in regular algebras, providing a generalisation of conditional probability distributions to general valuation or even generalised information algebras. For this aspect of regular valuation algebras we refer to (Kohlas, 2003a). Those results could be extended to generalised information algebras, but we shall not pursue this line of inquiry.

Further, we remark that the relation $\phi \leq_2 \psi$ if there is an idempotent f such that $\psi = f \cdot \phi$ is a *partial order*. Of course $\phi \leq_2 \psi$ implies $\phi \leq \psi$. This is the partial order studied in semigroup theory (Nambooripad, 1980; Mitsch, 1986), the goal there being to study the structure of semigroups. The condition $\psi = f \cdot \phi$ means in our context that ψ is obtained by combination of ϕ with a deterministic information f . So ψ results from a kind of conditioning of ϕ on f . We refer to (Kohlas, 2003a) for an illustration in the context of probability potentials. So, ψ is, according to this order, more informative than ϕ , if it is obtained by conditioning of ϕ . Although this makes sense, this order does not seem very interesting from the point of view of information algebra. For example it does not follow that $\epsilon_x(\psi) \leq \psi$.

In labeled algebras, the concept of regularity is similarly defined (Kohlas, 2003a). As an example of a regular labeled valuation algebra, we consider the case of the labeled valuation algebra of probability potentials.

Example 6.1 In Section 3.3 probability potentials were introduced as a semiring valuation algebra with values in the semiring $A = (\mathbb{R}^+ \cup \{0\}; +, \times)$. Probability potentials are mappings $p : \Theta \mapsto \mathbb{R}^+ \cup \{0\}$ from the frames of a f.c.f to nonnegative real numbers. This labeled valuation algebra is regular, in the sense that for any probability potential p on a frame Θ and $\Lambda \leq \Theta$ there is a potential q on Λ such that $p = p \cdot \pi_\Lambda(p) \cdot q$. In fact, the potential q is determined as follows

$$q(\lambda) = \begin{cases} \frac{1}{\pi_\Lambda(p)(\lambda)} & \text{if } \pi_\Lambda(p)(\lambda) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The idempotents of the group $[p]_\gamma$ of a potential p is the potential $f_p(\theta) = 1$ for all θ for which $p(\theta) > 0$ and $f_p(\theta) = 0$ for θ with $p(\theta) = 0$. So, the idempotents define the *support sets* $\{\theta : p(\theta) > 0\}$ of the probability potentials. Note that the projection of an idempotent is not itself an idempotent. The idempotent valuation algebra $(F, D; d, \cdot, \bar{\pi})$, defined similarly as in the domain-free case, corresponds to the labeled set algebra of subsets of the frames Θ . \ominus

6.3 Separative Algebras

Here we go one step beyond regular algebras. Consider again a domain-free generalised information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. Instead of assuming it to be regular, and then use the Green relation to study order, we start with a congruence, similar to the Green relation and base the study of order on this relation. Thus, assume that there is a congruence \equiv_γ relative to combination and extraction in Ψ such that

$$\epsilon_x(\psi) \cdot \psi \equiv_\gamma \psi \tag{6.9}$$

for all $\psi \in \Psi$ and $x \in D$. Since any element ψ has a support, we have also

$$\psi \cdot \psi \equiv_\gamma \psi$$

The equivalence classes $[\psi]_\gamma$ are semigroups. Indeed, if $\phi, \chi \in [\psi]_\gamma$, then $\phi \equiv_\gamma \chi$ and $\chi \equiv_\gamma \psi$, hence $\phi \cdot \chi \equiv_\gamma \psi \cdot \psi$ since \equiv_γ is a congruence. But $\psi \cdot \psi \equiv_\gamma \psi$, thus $\phi \cdot \chi \equiv_\gamma \psi$ hence $\phi \cdot \chi \in [\psi]_\gamma$.

As in the previous section the quotient algebra $(\Psi/\gamma, D; \leq, \perp, \cdot, \epsilon)$ is an idempotent information algebra, homomorphic to $(\Psi, D; \leq, \perp, \cdot, \epsilon)$, and the operations are defined as

1. *Combination*: $[\phi]_\gamma \cdot [\psi]_\gamma = [\phi \cdot \psi]_\gamma$.
2. *Extraction*: $\epsilon_x([\psi]_\gamma) = [\epsilon_x(\psi)]_\gamma$.

Idempotency of $(\Psi/\gamma, D; \leq, \perp, \cdot, \epsilon)$ follows from condition (6.9).

Since the classes form an idempotent algebra, they are partially ordered by $[\phi]_\gamma \leq [\psi]_\gamma$ if $[\phi]_\gamma \cdot [\psi]_\gamma = [\phi]_\gamma$. Under this order we have

$$[\phi]_\gamma \cdot [\psi]_\gamma = [\phi]_\gamma \vee [\psi]_\gamma.$$

As in the previous section, we would like this partial order of classes to represent the preorder defined in (6.1) in the sense that $\phi \leq \psi$ iff $[\phi]_\gamma \leq [\psi]_\gamma$. But for this to hold, we need a further condition, since the classes $[\psi]_\gamma$ are in general, in contrast to regular algebras, not groups.

In semigroup theory embeddings of semigroups into a disjoint union of groups is studied, see (Clifford & Preston, 1967). A sufficient condition for this to be possible is *cancellativity*, that is

$$\phi \cdot \psi = \phi \cdot \psi' \tag{6.10}$$

implies $\psi = \psi'$. We assume therefore that all semigroups $[\phi]_\gamma$ are cancellative. This leads to the following definition.

Definition 6.2 *Separative Information Algebras*: Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be a generalised domain-free information algebra. It is called *separative*, if there exists a congruence \equiv_γ relative to combination and extraction in Ψ such that

1. $\epsilon_x(\psi) \cdot \psi \equiv_\gamma \psi$ for all $\psi \in \Psi$ and for all $x \in D$.
2. The semigroups $[\psi]_\gamma$ are cancellative for all $\psi \in \Psi$.

We remark that separative valuation algebras have been studied in (Kohlas, 2003a) with respect to local computation with division and to generalisation of conditionals from probability to general valuations or information. As in the case of regular algebras, the theory of conditionals is closely related to natural order. Here we focus on information order in generalised information algebras. For examples of separative valuation algebras, we refer to

(Kohlas, 2003a; Pouly & Kohlas, 2011). We mention also, that as far as local computation with division and conditioning is concerned, it is sufficient that \equiv_γ is a congruence with respect to combination only. But for the present theory of order, congruence with respect to extraction is also desirable and many separative instances satisfy this condition.

A cancellative semigroup such as $[\psi]_\gamma$ can be embedded into a group. The classical procedure is like for embedding integers into rational numbers as follows: Consider ordered pairs (ϕ, ψ) for $\phi, \psi \in [\psi]_\gamma$ and define a relation among pairs by

$$(\phi, \psi) \equiv (\phi', \psi') \text{ iff } \phi \cdot \psi' = \phi' \cdot \psi.$$

This is an equivalence relation thanks to cancellativity. Let $[\phi, \psi]$ denote the equivalence classes of this equivalence and let $\gamma(\psi)$ denote the set of these pairs from $[\psi]_\gamma$. Then we define the operation

$$[\phi, \psi] \cdot [\phi', \psi'] = [\phi \cdot \phi', \psi \cdot \psi']$$

in $\gamma(\psi)$. This is well defined, since the equivalence is a congruence relative to the operation $(\phi, \psi) \cdot (\phi', \psi') = (\phi \cdot \phi', \psi \cdot \psi')$ between pairs. With this operation $\gamma(\psi)$ becomes a group. Its unit is $[\psi, \psi]$ and the inverse of $[\phi, \psi]$ is $[\psi, \phi]$. The class $[\psi]_\gamma$ is embedded into $\gamma(\psi)$ as a semigroup by the map

$$\psi \mapsto [\psi \cdot \psi, \psi].$$

Define

$$\Psi^* = \bigcup_{\psi \in \Psi} \gamma(\psi).$$

In order to distinguish elements of Ψ^* from those of Ψ , we denote elements of Ψ^* by lower case letters like a, b, \dots . The union of groups Ψ^* becomes a semigroup, if we define for $a = [\phi_a, \psi_a]$ and $b = [\phi_b, \psi_b]$,

$$a \cdot b = [\phi_a \cdot \phi_b, \psi_a \cdot \psi_b].$$

This operation is well-defined, associative and commutative. Thus $(\Psi^*; \cdot)$ is a commutative semigroup and $(\Psi; \cdot)$ is embedded into it as a semigroup by the map $\psi \mapsto [\psi \cdot \psi, \psi]$ as can easily be verified. In the sequel, in order to simplify notation, we denote the elements $[\psi \cdot \psi, \psi]$ of the image of $(\Psi; \cdot)$ under this map simply by ψ .

We may carry over the order between the classes $[\psi]_\gamma$ to the groups $\gamma(\psi)$, since there is a one-to-one relation between classes and groups. Hence $\gamma(\phi) \leq \gamma(\psi)$ iff $[\phi]_\gamma \leq [\psi]_\gamma$. Then we deduce that

$$\gamma(\phi \cdot \psi) = \gamma(\phi) \vee \gamma(\psi).$$

We extend now the natural order (6.1) to the semigroup $(\Psi^*; \cdot)$,

$$a \leq b, \text{ iff there exists a } c \in \Psi^* \text{ such that } b = a \cdot c. \quad (6.11)$$

Note that for elements of Ψ , this preorder $\phi \leq \psi$ admits that in $\psi = \phi \cdot c$, the factor which completes ϕ to ψ does no more need to be an element of Ψ , but only of Ψ^* .

Lemma 6.2 *In Ψ^* we have $a \leq b$ iff $\gamma(a) \leq \gamma(b)$.*

Proof. Assume first $a \leq b$, hence $a \cdot c = b$ for some $c \in \Psi^*$. Then $\gamma(b) = \gamma(a \cdot c) = \gamma(a) \vee \gamma(c)$, hence $\gamma(a) \leq \gamma(b)$. Conversely, assume $\gamma(a) \leq \gamma(b)$. Then $\gamma(b) = \gamma(a) \vee \gamma(b) = \gamma(a \cdot b)$. Therefore we see that $a \cdot b$ and b belong both to the group $\gamma(b)$ and therefore $b = a \cdot b \cdot (a \cdot b)^{-1} \cdot b$, thus $a \leq b$. \square

We remark that for any element a of Ψ^* we have $a = a \cdot a^{-1} \cdot a$. This means that the semigroup (Ψ^*, \cdot) is *regular*. And $a \equiv_\gamma b$ implies $a \cdot \Psi^* = b \cdot \Psi^*$. In fact, if $d \in a \cdot \Psi^*$, then $d = a \cdot c$ for some $c \in \Psi^*$. It follows then $d = b \cdot b^{-1} \cdot a \cdot c$, hence $d \in b \cdot \Psi^*$. In the same way it follows that $d \in b \cdot \Psi^*$ implies $d \in a \cdot \Psi^*$, hence $a \cdot \Psi^* = b \cdot \Psi^*$. Conversely, if $a \cdot \Psi^* = b \cdot \Psi^*$, then $a = b \cdot c$ and $b = a \cdot c'$ for some $c, c' \in \Psi^*$. This means that $a \leq b$ and $b \leq a$, hence $\gamma(a) = \gamma(b)$, or $a \equiv_\gamma b$. This shows that the congruence \equiv_γ is the Green relation in the regular semigroup (Ψ^*, \cdot) .

As a consequence of this remark and of Lemma 6.2 we have, as in the previous section (Lemma 6.1), the following result:

Lemma 6.3 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be a separative information algebra. Then*

1. $a \leq b$ and $b \leq a$ iff $\gamma(a) = \gamma(b)$,
2. $\phi \leq \psi$ iff $[\phi]_\gamma \leq [\psi]_\gamma$,
3. $\phi \leq \psi$ and $\psi \leq \phi$ iff $[\phi]_\gamma = [\psi]_\gamma$.

As in the case of regular information algebras, we have for separative information algebras the same results regarding order and extraction (see Theorem 6.2).

Theorem 6.4 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be a separative generalised information algebra. Then*

1. $\epsilon_x(\psi) \leq \psi$ for all $x \in D$ and $\psi \in \Psi$.
2. $\phi \leq \psi$ implies $\epsilon_x(\phi) \leq \epsilon_x(\psi)$ for all $x \in D$.
3. $x \leq y$ implies $\epsilon_x(\psi) \leq \epsilon_y(\psi)$ for all $\psi \in \Psi$.

Proof. 1.) From (6.9) we obtain $\gamma(\epsilon_x(\psi) \cdot \psi) = \gamma(\epsilon_x(\psi)) \vee \gamma(\psi) = \gamma(\psi)$. this shows that $\gamma(\epsilon_x(\psi)) \leq \gamma(\psi)$, which implies $\epsilon_x(\psi) \leq \psi$ (Lemma 6.2).

2.) From $\phi \leq \psi$ we obtain $\gamma(\phi) \leq \gamma(\psi)$ and from item 1 just proved $\gamma(\epsilon_x(\phi)) \leq \gamma(\phi)$. Thus we have $\gamma(\epsilon_x(\phi) \cdot \psi) = \gamma(\epsilon_x(\phi)) \vee \gamma(\psi) = \gamma(\psi)$. Further, we have $\epsilon_x(\epsilon_x(\phi) \cdot \psi) = \epsilon_x(\phi) \cdot \epsilon_x(\psi)$. Therefore, from the congruence of \equiv_γ , we conclude that $\gamma(\epsilon_x(\phi) \cdot \epsilon_x(\psi)) = \gamma(\epsilon_x(\psi))$, and this shows that $\epsilon_x(\phi) \leq \epsilon_x(\psi)$.

3.) This is proved exactly as item 3 of Theorem 6.2. \square

If $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is a separative generalised information algebra, then the quotient algebra $(\Psi/\gamma, D; \leq, \perp, \cdot, \epsilon)$, is an idempotent information algebra, homomorphic to $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ as noted above. Any group γ_ψ has a unique unit and idempotent element, denoted by f_ψ . The idempotent information algebra of idempotents or the units of the groups $\gamma(\psi)$, $(F, D; \leq, \perp, \cdot, \bar{\epsilon})$, with the operations defined as follows

1. *Combination:* $f_\phi \cdot f_\psi = f_{\phi \cdot \psi}$,
2. *Extraction:* $\bar{\epsilon}_x(f_\psi) = f_{\epsilon_x(\psi)}$,

is isomorphic to the quotient algebra $(\Psi/\gamma, D; \leq, \perp, \cdot, \epsilon)$. Note however, that the elements of F do not, in general, belong to Ψ . Nevertheless, we may still consider the elements of F as the deterministic parts of Ψ^* .

As in the regular case, we may define an order $\phi \leq_2 \psi$ if there is an idempotent f such that $\psi = f \cdot \phi$ and again $\phi \leq_2 \psi$ implies $\phi \leq \psi$. This is as before a partial order, since $\phi \leq \psi$ and $\psi \leq \phi$ imply $\gamma(\phi) = \gamma(\psi)$ and $\psi = f_\psi \cdot \phi$. But $f_\psi = f_\phi$, hence $\psi = f_\phi \cdot \phi = \phi$. The expression $f_\psi \cdot \phi$ is again a kind of conditioning, namely the combination of a deterministic element f_ψ with an information element ϕ . We refer to (Kohlas, 2003a) for a discussion of the separative valuation algebra of densities, which illustrates these statements. Again, it makes sense that an information ψ obtained from another one by condition $\psi = f_\psi \cdot \phi$, where $f_\phi \leq f_\psi$ is considered to be more informative. At least in probability theory this seems evident.

Chapter 7

Proper Information

7.1 Ideal Completion

In this section we consider domain-free *idempotent* generalised information algebras $(\Psi, D; \leq, \perp, \cdot, \epsilon)$, that is, *proper* information algebras. Their associated idempotent valuation were called information algebras in (Kohlas, 2003a). There it was shown that the partial order introduced by idempotency plays an important role in the theory of idempotent information algebras, see also (Kohlas & Schmid, 2014; Kohlas & Schmid, 2016). Many of these results carry over to the more general case of idempotent generalised information algebras. Some of them will be presented here.

In an idempotent information algebra, we have defined $\phi \leq \psi$ if $\phi \cdot \psi = \psi$, see Section 6.1. So, ψ is more informative than ϕ , if adding ϕ to ψ gives nothing new. Another way to express this is to say the piece of information ϕ is contained in ψ or also the piece of information ϕ is implied by ψ . As a consequence, combination corresponds to the supremum in this order, $\phi \cdot \psi = \phi \vee \psi$. So the idempotent semigroup (Ψ, \cdot) determines a join-semilattice $(\Psi; \leq)$. If we want to stress the point of view of order we write $\phi \vee \psi$ instead of $\phi \cdot \psi$.

Instead of looking at a particular piece of information ψ we may look at families of pieces of information I . Such a family is consistent and complete if for any $\psi \in I$, all elements less informative, implied by ψ , belong also to I , and whenever ϕ and ψ belong to I , then the combined information $\phi \cdot \psi$ belongs to I . This means that I is an *ideal* in the semilattice $(\Psi; \leq)$. More formally, I is an ideal if

1. $\phi \leq \psi$ and $\psi \in I$ imply $\phi \in I$,
2. $\phi, \psi \in I$ imply $\phi \vee \psi \in I$.

The set $\downarrow \psi$ of all elements less informative than or implied by ψ form an ideal, a *principal ideal*. Note that the unit is in all ideals, and if ψ belongs to an ideal, then all extractions $\epsilon_x(\psi)$ belong to the ideal also. The null

element 0 belongs only to the ideal Ψ . All ideals different from Ψ are called *proper ideals*.

An ideal can also be seen as information. In fact, we may extend the operations of combination and extraction from the algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ to the set of ideals I_Ψ of it:

1. *Combination*:

$$I_1 \cdot I_2 = \{\psi \in \Psi : \exists \psi_1 \in I_1, \psi_2 \in I_2 \text{ such that } \psi \leq \psi_1 \cdot \psi_2\}. \quad (7.1)$$

2. *Extraction*:

$$\epsilon_x(I) = \{\psi \in \Psi : \exists \phi \in I \text{ such that } \psi \leq \epsilon_x(\phi)\}. \quad (7.2)$$

These operation are well-defined, since they yield ideals in both cases.

It turns out that the set of ideals I_Ψ of Ψ with these operations in fact becomes an information algebra. In order to show this, we need some preparations. First, the intersection of an arbitrary family of ideals is still an ideal. Therefore, the ideal generated by a subset X of Ψ can be defined as the smallest ideal containing X ,

$$I(X) = \bigcap \{I : I \in I_\Psi, X \subseteq I\}.$$

Alternatively, we have also

$$I(X) = \{\psi \in \Psi : \exists \psi_1, \dots, \psi_n \in X \text{ such that } \psi \leq \psi_1 \cdot \dots \cdot \psi_n\}. \quad (7.3)$$

In particular, we have $I_1 \cdot I_2 = I(I_1 \cup I_2)$. If X is a finite subset of Ψ , then

$$I(X) = \downarrow \vee X,$$

the ideal generated by X is the principal ideal of the element $\vee X \in \Psi$. These are well-known result, (Kohlas, 2003a)

From lattice theory we know that a system closed under arbitrary intersections, a so-called \cap -system, forms a complete lattice under the partial order of inclusion, see (Davey & Priestley, 2002). Infimum is intersection and supremum is given by

$$\bigvee Y = \bigcap \{I : I \in I_\Psi, \bigcup_{J \in Y} J \subseteq I\}$$

where Y is any family of ideals. In particular, we have $I_1 \cdot I_2 = I_1 \vee I_2$.

For I_Ψ to become an information algebra in the sense of Chapt. 5, any ideal must have a support $x \in D$, that is $\epsilon_x(I) = I$. But if the semilattice D does not have a greatest element, this may not hold. If D has a greatest element \top , then by the support axiom for the information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ this element is necessarily a support for every element of Ψ . In

this case any ideal of Ψ has at least this element as support. If D has no greatest element, we may adjoin one in the following way: Consider $(D \cup \{\top\}, \leq, \perp)$ where $x \leq \top$ and $x \vee \top = \top$ for all elements x of $D \cup \{\top\}$. Extend the conditional independence relation with $x \perp y | \top$ for all x and y in $D \cup \{\top\}$. Then $(D \cup \{\top\}; \leq, \perp)$ is still a q-separoid. Further, define ϵ_\top as the identity map of Ψ . Then, clearly, $(\Psi, D \cup \{\top\}; \leq, \perp, \cdot, \epsilon)$ is still an idempotent generalised information algebra; in particular the axioms of extraction, combination and idempotency are still valid in this extended structure. Therefore, we assume in the sequel, without loss of generality, that D has a top element. Then $(I_\Psi, D; \leq, \perp, \cdot, \epsilon)$ becomes an idempotent generalised information algebra.

Theorem 7.1 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be an idempotent generalised domain-free information algebra such that $(D; \leq)$ has a greatest element. Then $(I_\Psi, D; \leq, \perp, \cdot, \epsilon)$ is an idempotent generalised domain-free information algebra and $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is embedded into it.*

Proof. Axiom A0 holds, since $(D; \leq, \perp)$ is the same q-separoid as in $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. Axiom A1 (Semigroup) holds, since Ψ is under the order induced by combination a complete lattice. Axiom A2 (Support) follows since the greatest element of D is a support of any ideal I . Further, we must show that if $\epsilon_x(I) = I$ and $y \geq x$, then $\epsilon_y(I) = I$. Remark that if $\psi \in \epsilon_x(I) = I$, then $\psi \leq \epsilon_x(\phi) = \eta$ for some $\phi \in I$. Note that η has support x and belongs to I . So $\psi \leq \eta = \epsilon_x(\eta)$, $\eta \in I$. Therefore, if $\epsilon_x(I) = I$, then any element ψ of I is dominated by an element with support x from I . Now, clearly $\epsilon_y(I) \subseteq I$. Consider then an element ψ from I . By the preceding remark there is an element η in I with support x , such that $\psi \leq \eta$. By the support axiom for the algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$, $y \geq x$ is then also a support for η . So, we have $\psi \leq \epsilon_y(\eta)$. This shows that $\psi \in \epsilon_y(I)$, hence $\epsilon_y(I) = I$. This verifies Axiom A2 for the algebra $(I_\Psi, D; \leq, \perp, \cdot, \epsilon)$. Unit in $(I_\Psi; \cdot)$ is the principal ideal $\{1\}$ and null element is Ψ . It is obvious that the Unit and Null Axiom A3 holds.

To verify the Extraction Axiom A4, we must show that $x \perp y | z$ and $\epsilon_x(I) = I$ implies $\epsilon_y(I) = \epsilon_y(\epsilon_z(I))$. Now, if $\psi \in \epsilon_y(\epsilon_z(I))$, then $\psi \leq \epsilon_y(\phi)$ for some element ϕ such that $\phi \leq \epsilon_z(\eta)$ for an element η in I . Then $\psi \leq \epsilon_y(\epsilon_z(\eta)) \leq \epsilon_y(\eta)$. This shows that $\epsilon_y(\epsilon_z(I)) \subseteq \epsilon_y(I)$. Consider then an element $\psi \in \epsilon_y(I)$. There is an element $\eta' \in I$ such that $\psi \leq \epsilon_y(\eta')$. But $\eta' \in I = \epsilon_x(I)$ implies that there is also an element $\eta \in I$ with support x such that $\eta' \leq \eta$. Therefore $\psi \leq \epsilon_y(\eta)$ and $\eta = \epsilon_x(\eta)$. By the Extraction Axiom A4 for the algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$, we have $\epsilon_y(\eta) = \epsilon_y(\epsilon_z(\eta))$. Since $\epsilon_z(\eta)$ belongs to $\epsilon_z(I)$ this implies that $\psi \in \epsilon_y(\epsilon_z(I))$, therefore $\epsilon_y(I) = \epsilon_y(\epsilon_z(I))$ and Axiom A4 holds in the algebra $(I_\Psi, D; \leq, \perp, \cdot, \epsilon)$ too.

For the Combination Axiom A5, assume $x \perp y | z$ and $\epsilon_x(I_1) = I_1$, $\epsilon_y(I_2) = I_2$. Then we must show that $\epsilon_z(I_1 \cdot I_2) = \epsilon_z(I_1) \cdot \epsilon_z(I_2)$. Consider an element ψ

from $\epsilon_z(I_1 \cdot I_2)$, such that $\psi \leq \epsilon_z(\phi)$ for a $\phi \in I_1 \cdot I_2$. Then there are elements $\phi_1 = \epsilon_x(\phi_1) \in I_1$ and $\phi_2 = \epsilon_y(\phi_2) \in I_2$ such that $\phi \leq \phi_1 \cdot \phi_2$. This implies $\psi \leq \epsilon_z(\phi_1 \cdot \phi_2)$. By Axiom A4 for the algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ we obtain then $\psi \leq \epsilon_z(\phi_1 \cdot \phi_2) = \epsilon_z(\phi_1) \cdot \epsilon_z(\phi_2)$, which shows that $\psi \in \epsilon_z(I_1) \cdot \epsilon_z(I_2)$. Conversely, consider $\psi \in \epsilon_z(I_1) \cdot \epsilon_z(I_2)$. Then $\psi \leq \psi_1 \cdot \psi_2$, where $\psi_1 \leq \epsilon_z(\phi_1)$ and $\phi_1 = \epsilon_x(\phi_1)$, $\phi_1 \in I_1$, and, similarly, $\psi_2 \leq \epsilon_z(\phi_2)$, with $\phi_2 = \epsilon_y(\phi_2)$ and $\phi_2 \in I_2$. So we have $\psi \leq \epsilon_z(\phi_1) \cdot \epsilon_z(\phi_2)$ and by axiom A4 for the algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ the left hand side equals $\epsilon_z(\phi_1 \cdot \phi_2)$. So $\psi \leq \epsilon_z(\phi_1 \cdot \phi_2)$ with $\phi_1 \in I_1$ and $\phi_2 \in I_2$. This shows that $\psi \in \epsilon_z(I_1 \cdot I_2)$. Thus we obtain that $\epsilon_z(I_1 \cdot I_2) = \epsilon_z(I_1) \cdot \epsilon_z(I_2)$.

Idempotency, Axiom A6, follows from $\epsilon_x(I) \subseteq I$, hence $\epsilon_x(I) \cdot I = \epsilon_x(I) \vee I = I$.

This proves that $(I_\Psi, D; \leq, \perp, \cdot, \epsilon)$ is an idempotent generalised information algebra. Consider now the pair of maps (f, g) defined by $f(\psi) = \downarrow \psi$ and $g(\epsilon_x) = \epsilon_x$ (on the left ϵ_x as extraction in Ψ , on the right as extraction in I_Ψ). Then

$$\begin{aligned} f(\phi \cdot \psi) &= \downarrow(\phi \cdot \psi) = \downarrow \phi \cdot \downarrow \psi, \\ f(\epsilon_x(\psi)) &= \downarrow \epsilon_x(\psi) = \epsilon_x(\downarrow \psi). \end{aligned} \tag{7.4}$$

These identities follow directly from the definition of combination and extraction in I_Ψ . Further we have $f(1) = \{1\}$ and $f(0) = \Psi$. So, the pair of maps (f, g) is a homomorphism. It is also one-to-one, since $\downarrow \phi = \downarrow \psi$ imply $\phi = \psi$. So (f, g) is an embedding of $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ into $(I_\Psi, D; \leq, \perp, \cdot, \epsilon)$. This completes the proof. \square

Ideal completion of an idempotent valuation algebra gives an idempotent valuation algebra. This case has already been discussed in (Kohlas, 2003a). From a lattice theoretic point of view, ideal completion completes the join-semilattice (Ψ, \leq) to a complete lattice. In terms of combination or aggregation of information, completion means that any set X of pieces of information in Ψ can be aggregated, namely to the ideal $I(X)$ generated by X . A different approach to completion in idempotent valuation algebras has been proposed in (Guan & Li, 2010).

7.2 Compact Algebras

In information processing, only “finite” information can be handled. “Infinite” information can however often be approximated by “finite” elements. This aspect of finiteness is discussed in this section. It must be stressed that not every aspect of finiteness is captured. For example, no questions of computability and related issues will be treated. On the other hand, many aspects discussed in this section, are also considered in domain theory. In fact, much of this section is motivated by domain theory. However, the one crucial feature not addressed in domain theory, is information extraction.

Also domain theory places more emphasis on order, approximation and convergence of information and less on combination. So, although the subject is similar to domain theory, it is treated here with a different emphasis and goal. It will be seen that the subject is also closely related to *algebraic* and, more generally, *continuous lattices*.

Consider an idempotent generalised information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. In the set of pieces of information Ψ we single below out a subset Ψ_f of elements which are considered to be finite. In fact, we adopt the concept of finiteness in order theory. For this purpose, we need the concept of a directed set in Ψ : A subset X of Ψ is called *directed*, if it is not empty and with any two elements ϕ_1 and ϕ_2 belonging to it, there is an element $\phi \in X$ such that $\phi_1, \phi_2 \leq \phi$. Then finite elements are defined as follows (Davey & Priestley, 1990):

Definition 7.1 *Finite Elements*: *An element ψ in $(\Psi; \leq)$ is called finite or also compact, if for any directed set X in Ψ whose supremum exists in Ψ , from $\psi \leq \vee X$ it follows that there is an element $\phi \in X$ such that $\psi \leq \phi$.*

Let Ψ_f denote the set of finite elements of $(\Psi; \leq)$. This set of finite elements is closed under combination. Indeed, if ϕ and ψ are finite and $\phi \cdot \psi \leq \vee X$ for some directed subset X of Ψ , then from $\phi, \psi \leq \phi \cdot \psi$ it follows that there are elements ϕ' and ψ' in X such that $\phi \leq \phi'$ and $\psi \leq \psi'$. Further, since X is directed there is an element χ in X such that $\phi', \psi' \leq \chi$, hence $\phi \cdot \psi \leq \phi' \cdot \psi' \leq \chi \in X$. This shows that $\phi \cdot \psi$ is finite. Further the unit 1 is clearly finite. One might expect that extraction of finite information always results in finite information. Although this is the case for many instances, it is not true in general.

We want now to consider information algebras, where every element can be approximated by the finite elements it dominates, that is, the finite elements, which are less informative. In fact, we want even more: An element with support x must be approximated by the finite elements supported by the same domain x . This is captured by the following definition.

Definition 7.2 *Compact Information Algebra*: *An idempotent, domain-free generalised information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is called compact, if for all $\phi \in \Psi$ and $x \in D$, if $\phi = \epsilon_x(\phi)$,*

$$\phi = \epsilon_x(\phi) = \bigvee \{\psi \in \Psi_f, \psi = \epsilon_x(\psi) \leq \phi\}. \quad (7.5)$$

Identity (7.5) means that the finite elements of domain x are dense in this domain, finite elements approximate all elements of this domain. This is called *strong density*. It implies also that the whole set of finite elements is dense in Ψ , in the sense that they approximate any element ϕ of Ψ . In

fact, such an element has some support x (by the Support Axiom), hence by strong density,

$$\begin{aligned}\phi &= \epsilon_x(\phi) = \bigvee \{\psi \in \Psi_f, \psi = \epsilon_x(\psi) \leq \phi\} \\ &\leq \bigvee \{\psi \in \Psi_f, \psi \leq \phi\} \leq \phi.\end{aligned}$$

Thus, we see that

$$\phi = \bigvee \{\psi \in \Psi_f, \psi \leq \phi\}. \quad (7.6)$$

This is also called *weak density*. Note that weak density does not imply strong density. A counter example for this is given in (Kohlas, 2003a).

Here is a first result, which expresses a continuity property of the extraction operators ϵ_x .

Theorem 7.2 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be a compact information algebra. If X is a directed subset of Ψ such that $\bigvee X$ exists in Ψ , and $x \in D$, then $\bigvee_{\phi \in X} \epsilon_x(\phi)$ exists in Ψ and*

$$\epsilon_x(\bigvee X) = \bigvee_{\phi \in X} \epsilon_x(\phi). \quad (7.7)$$

Proof. If $\phi \in X$, then $\phi \leq \bigvee X$, hence $\epsilon_x(\phi) \leq \epsilon_x(\bigvee X)$ and therefore $\epsilon_x(\bigvee X)$ is an upper bound of the $\epsilon_x(\phi)$ for $\phi \in X$.

By density,

$$\epsilon_x(\bigvee X) = \bigvee \{\psi \in \Psi_f : \psi = \epsilon_x(\psi) \leq \bigvee X\}.$$

But $\psi \leq \bigvee X$ implies that there is a $\phi \in X$ such that $\psi \leq \phi$. Then $\psi = \epsilon_x(\psi) \leq \epsilon_x(\phi)$, hence $\epsilon_x(\bigvee X)$ is the least upper bound of the $\epsilon_x(\phi)$ for $\phi \in X$, therefore $\epsilon_x(\bigvee X) = \bigvee_{\phi \in X} \epsilon_x(\phi)$. \square

In general, not every directed set in Ψ has a supremum in Ψ . But if this is the case, i.e. if $(\Psi; \leq)$ is a directed complete partial order (dcpo), then, since Ψ is closed under finite joins, it follows by standard results from lattice theory, that $(\Psi; \leq)$ is a complete lattice. If $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is compact, then weak density (7.6) holds in Ψ . A complete lattice $(\Psi; \leq)$ with the additional condition (7.6) is called an *algebraic lattice*, (Gierz, 2003). Therefore, we call a compact information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$, in which $(\Psi; \leq)$ is a dcpo, an *algebraic information algebra*.

Definition 7.3 *A compact information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is called algebraic, if $(\Psi; \leq)$ is a dcpo (directed complete partial order).*

In summary, an algebraic information algebra is an algebraic lattice in which strong density holds; and conversely, an information algebra which is

an algebraic lattice in which strong density holds, is an algebraic information algebra. In (Kohlas, 2003a), an alternative, but equivalent definition of an algebraic information algebra has been given according to the following theorem.

Theorem 7.3 *An idempotent, domain-free generalised information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is an algebraic information algebra, if and only if there exists a subset Ψ' of Ψ , closed under combination and containing the unit 1, satisfying the following conditions:*

1. *Convergence:* For any directed subset X of Ψ' , the supremum $\vee X$ exists in Ψ .
2. *Density:* For any $\phi \in \Psi$ and $x \in D$ such that $\phi = \epsilon_x(\phi)$,

$$\phi = \epsilon_x(\phi) = \bigvee \{\psi \in \Psi', \psi = \epsilon_x(\psi) \leq \phi\}. \quad (7.8)$$

3. *Compactness:* For any directed subset X of Ψ' and $\psi \in \Psi'$ such that $\psi \leq \bigvee X$, there is an element $\phi \in X$ such that $\psi \leq \phi$.

Then Ψ' equals the set of finite elements Ψ_f of $(\Psi; \leq)$.

The full proof of this theorem can be found in (Kohlas, 2003a). Although it is given there for idempotent valuation algebras, the proof carries over to the more general case of idempotent information algebras. Item 1 in the conditions above is sufficient to make $(\Psi; \leq)$ a complete lattice; item 3 is then equivalent to the finiteness condition in $(\Psi; \leq)$, such that $\Psi' = \Psi_f$ and item 2 becomes strong density. The algebra is therefore compact, hence algebraic.

An algebraic information algebra may be obtained from any information algebra by ideal completion.

Theorem 7.4 *If $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is an idempotent generalised information algebra such that $(D; \leq)$ has a greatest element, then its ideal completion $(I_\Psi, D; \leq, \perp, \cdot, \epsilon)$ is an algebraic information algebra with the set $\{\downarrow \phi : \phi \in \Phi\}$ of principal ideals as finite elements.*

This has already been shown in (Kohlas, 2003a) for idempotent valuation algebras. Since it has been shown in Theorem 7.1 in the previous section that $(I_\Psi, D; \leq, \perp, \cdot, \epsilon)$ is an idempotent generalised information algebra, the rest of the proof does not change for idempotent generalised information algebras, since only convergence, density and compactness in Theorem 7.3 need to be verified; therefore we refer to (Kohlas, 2003a) for a proof.

The algebraic information algebra $(I_\Psi, D; \leq, \perp, \cdot, \epsilon)$ is fully determined by its finite elements, that is, the elements of Ψ . This holds in general

for any compact or algebraic information algebra, as the following theorems shows. This is similar to a well-known result in domain theory (Stoltenberg-Hansen *et al.*, 1994). To show that in a compact algebra the elements are fully determined by their finite elements, there are at least two approaches possible. One is ideal completion of the partial order (Ψ_f, \leq) of finite elements, the other adds suprema of directed sets of finite elements which have no suprema in Ψ (Guan & Li, 2010). It seems that for the second approach the finite elements must be closed under extraction, whereas for ideal completion this is not necessary. Otherwise the two approaches yield equivalent results.

We prove first the ideal representation theorem for algebraic information algebras, which shows that an algebraic information algebra is determined by its finite elements through ideal completion.

Theorem 7.5 *Assume $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ to be a domain-free algebraic information algebra, where $(D; \leq)$ has a greatest element \top . If Ψ_f are the finite elements of the algebraic information algebra, then the ideal completion $(I_{\Psi_f}, D; \leq, \perp, \cdot, \epsilon)$ may be extended to an algebraic information algebra, isomorphic to $(\Psi, D; \leq, \perp, \cdot, \epsilon)$.*

Proof. If Ψ_f is closed under extraction, then $\Psi_f \cup \{0\}$ forms a subalgebra of $(\Psi, D; \leq, \perp, \cdot, \epsilon)$, and it follows in this case from Theorem 7.1 that $(I_{\Psi_f}, D; \leq, \perp, \cdot, \epsilon)$ is an idempotent generalised information algebra. But we do not assume that the finite elements are closed under extraction, so we must first show that from I_{Ψ_f} and D we may nevertheless construct an idempotent information algebra.

We define first combination between ideals I_1 and I_2 of Ψ_f as usual,

$$I_1 \cdot I_2 = \{\psi \in \Psi_f : \exists \psi_1 \in I_1, \psi_2 \in I_2 \text{ such that } \psi \leq \psi_1 \cdot \psi_2\}.$$

As before, the ideals in Ψ_f form a \cap -system, hence a complete lattice, with combination as join. So, $(I_{\Psi_f}; \cdot)$ is a commutative semigroup with unit $\{1\}$ and null element Ψ_f . Hence the Semigroup Axiom A1 of a domain-free information algebra is satisfied.

Next, for any $x \in D$ we define an extraction operator

$$\epsilon_x(I) = \{\psi \in \Psi_f : \exists \phi \in I \text{ such that } \psi \leq \epsilon_x(\phi)\}.$$

Clearly, $\epsilon_x(I)$ is still an ideal in Ψ_f and ϵ_x maps therefore I_{Ψ_f} into itself.

The greatest element \top in D is a support for all elements of Ψ_f , hence we have $\epsilon_{\top}(I) = I$ for every ideal in Ψ_f . Thus, every element I of I_{Ψ_f} has a support. Assume further that x is a support for the ideal I of Ψ_f , that is $\epsilon_x(I) = I$ and $x \leq y$. Note that $\epsilon_y(I) \subseteq I$ and assume $\psi \in I$. Then $\psi \leq \epsilon_x(\phi)$ for some $\phi \in I$. But $\epsilon_x(\phi) \leq \epsilon_y(\phi)$. This implies $\psi \in \epsilon_y(I)$ and therefore $\epsilon_y(I) = I$. This confirms the Support Axiom A2 in $(I_{\Psi_f}, D; \leq, \perp, \cdot, \epsilon)$. The

Unit and Null Axiom A3 is obvious for ideals in Ψ_f . Extraction Axiom A4 and Combination Axiom A5 are proved just as in Theorem 7.1. The Idempotency Axiom A6 follows since $\epsilon_x(I) \subseteq I$. Axiom A0 is inherited from the algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. This shows that $(I_{\Psi_f}, D; \leq, \perp, \cdot, \epsilon)$ is a domain-free idempotent information algebra.

Consider now the map $\phi \mapsto A_\phi = \{\psi \in \Psi_f : \psi \leq \phi\}$. Since A_ϕ is an ideal in Ψ_f , this maps Ψ into I_{Ψ_f} . Consider any ideal I in Ψ_f . Then the supremum of I exists in Ψ , since the information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is algebraic. Let $\phi = \bigvee I$ and consider an element $\psi \in \Psi_f$ such that $\psi \leq \phi$. Then, by compactness, there is an element $\chi \in I$ dominating ψ . This implies $\psi \in I$ and this shows that $I = A_\phi$. So, the map is onto I_{Ψ_f} . It is one-to-one, since $A_\phi = A_\psi$ implies $\phi = \psi$.

It remains to show that the map is a homomorphism. Clearly $A_1 = \{1\}$, which is the unit element, and $A_0 = \Psi_f$, which is the null element of the ideal completion. Consider two elements ϕ and ψ from Ψ . Then, $A_{\phi \cdot \psi}$ contains both A_ϕ and A_ψ and $A_\phi \cdot A_\psi = I(A_\phi \cup A_\psi) \subseteq A_{\phi \cdot \psi}$. If I is an ideal in Ψ_f , which contains both A_ϕ and A_ψ , then there is an element $\chi \in \Psi$ such that $I = A_\chi$ and $\phi, \psi \leq \chi$, hence $A_{\phi \cdot \psi} \subseteq I$. Thus we conclude that $A_{\phi \cdot \psi} = A_\phi \cdot A_\psi$.

Further, for any x in D , we have by definition

$$\epsilon_x(A_\phi) = \{\psi \in \Psi_f : \exists \chi \in A_\phi \text{ such that } \psi \leq \epsilon_x(\chi)\}.$$

So, since $\epsilon_x(\chi) \leq \epsilon_x(\phi)$, it follows that $\epsilon_x(A_\phi) \subseteq A_{\epsilon_x(\phi)}$. Consider then an element ψ in $A_{\epsilon_x(\phi)}$. From $\phi = \bigvee A_\phi$, and from Theorem 7.2 we conclude that

$$\epsilon_x(\phi) = \bigvee A_{\epsilon_x(\phi)} = \bigvee_{\chi \in A_\phi} \epsilon_x(\chi).$$

The set $X = \{\epsilon_x(\chi) : \chi \in A_\phi\}$ is directed. By compactness, there is then an element $\eta \in A_\phi$ such that $\psi \leq \epsilon_x(\eta)$. But this means that $\psi \in \epsilon_x(A_\phi)$. Therefore, we see that $A_{\epsilon_x(\phi)} = \epsilon_x(A_\phi)$, which shows that the map $\phi \mapsto A_\phi$ is an information algebra homomorphism between $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ and $(I_{\Psi_f}, D; \leq, \perp, \cdot, \epsilon)$. This concludes the proof \square

This is a representation theorem for *algebraic* information algebras. What can be said about *compact* algebras? We first show, that a compact algebra can be extended to an algebraic one by adding the missing suprema of directed sets. Let then $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be a compact information algebra with finite elements Ψ_f . We assume now that the Ψ_f is closed under extraction. The approach used here follows (Guan & Li, 2010). Denote by Di_f the family of directed sets of $\Psi_f \cup \{0\}$. For such directed sets X and Y in Di_f , we define

$$\begin{aligned} X \cdot Y &= \{\phi \cdot \psi : \phi \in X, \psi \in Y\}, \\ \epsilon_x(X) &= \{\epsilon_x(\phi) : \phi \in X\}. \end{aligned} \tag{7.9}$$

These operations yield again directed sets of finite elements:

Lemma 7.1 *If $X, Y \in Dif$, then $X \cdot Y \in Dif$ and $\epsilon_x(X) \in Dif$ for all $x \in D$.*

Proof. Consider two elements η_1 and η_2 in $X \cdot Y$, such that $\eta_1 = \phi_1 \cdot \psi_1$ and $\eta_2 = \phi_2 \cdot \psi_2$ with $\phi_1, \phi_2 \in X$ and $\psi_1, \psi_2 \in Y$. Since X and Y are directed, there are elements $\phi \in X$ and $\psi \in Y$ such that $\phi_1, \phi_2 \leq \phi$ and $\psi_1, \psi_2 \leq \psi$. But then $\eta_1, \eta_2 \leq \phi \cdot \psi \in X \cdot Y$, which shows that $X \cdot Y$ is directed.

Similarly, consider $\phi_1, \phi_2 \in \epsilon_x(X)$, that is $\phi_1 = \epsilon_x(\psi_1)$ and $\phi_2 = \epsilon_x(\psi_2)$, where ψ_1 and ψ_2 belong to X . As X is directed, there is an element $\psi \in X$ which dominates ψ_1 and ψ_2 . But then it follows that $\phi_1, \phi_2 \leq \epsilon_x(\psi) \in \epsilon_x(X)$. This proves that $\epsilon_x(X)$ is directed and belongs to Dif , since we have assumed that the finite elements Ψ_f are closed under extraction. \square

Next, we define for directed sets X and Y in Dif the relation $X \equiv_\theta Y$ which holds if

- a) for all $\phi \in X$ there is a $\psi \in Y$ such that $\phi \leq \psi$,
- b) for all $\psi \in Y$ there is a $\phi \in X$ such that $\psi \leq \phi$.

This is an equivalence relation in Dif . Note that if X has a supremum in Ψ , then $X \equiv_\theta Y$ if and only if $\bigvee X = \bigvee Y$. Further, for any $\psi \in \Psi_f$, if $X \equiv_\theta \{\psi\}$, then necessarily $\psi \in X$ and $\bigvee X = \psi$. Now, in Dif/θ we define two operations between equivalence classes

- 1. *Combination:* $[X]_\theta \cdot [Y]_\theta = [X \cdot Y]_\theta$,
- 2. *Extraction:* $\epsilon_x([X]_\theta) = [\epsilon_x(X)]_\theta$.

These operations are well defined because \equiv_θ is a congruence relation for combination and extraction. Indeed assume $X_1 \equiv_\theta X_2$ and consider $\eta_1 \in X_1 \cdot Y$. Then $\eta_1 = \phi_1 \cdot \psi$ with $\phi_1 \in X_1$ and $\psi \in Y$. There is a $\phi_2 \in X_2$ such that $\phi_1 \leq \phi_2$, hence $\eta_1 \leq \phi_2 \cdot \psi \in X_2 \cdot Y$. In the same way, we find that for $\eta_2 \in X_2 \cdot Y$ there is an element in $X_1 \cdot Y$ which dominates η_2 . Therefore, we see that $X_1 \cdot Y \equiv_\theta X_2 \cdot Y$. Further, assume $X \equiv_\theta Y$ and consider an element ϕ from X . Then $\epsilon_x(\phi) \in \epsilon_x(X)$ and there is a $\psi \in Y$ such that $\phi \leq \psi$. But then $\epsilon_x(\phi) \leq \epsilon_x(\psi) \in \epsilon_x(Y)$. In the same way, if $\epsilon_x(\psi) \in \epsilon_x(Y)$, there is an element $\epsilon_x(\phi)$ in $\epsilon_x(X)$ such that $\epsilon_x(\psi) \leq \epsilon_x(\phi)$. Thus, $\epsilon_x(X) \equiv_\theta \epsilon_x(Y)$.

With these operations, it turns out that Dif becomes an idempotent algebraic information algebra, into which the original compact algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is embedded.

Theorem 7.6 *Assume $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ to be a domain-free compact information algebra, where $(D; \leq)$ has a greatest element \top and the set Ψ_f of*

finite elements is closed under extraction. Then $(Di_f/\theta, D; \leq, \perp, \cdot, \epsilon)$ is an algebraic information algebra and $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is embedded into it by the map

$$\phi \mapsto [\{\psi \in \Psi_f : \psi \leq \phi\}]_\theta.$$

Proof. We show first, that $(Di_f, D; \leq, \perp, \cdot, \epsilon)$ is a generalised, idempotent information algebra. Axiom A0, q-separoid, is valid, since $(D; \leq, \perp)$ is the same as in $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. Commutativity and Associativity of combination follow from commutativity and associativity of the operation $X \cdot Y$. The class $[\{1\}]_\theta$ is the unit and $[\Psi_f]_\theta$ is the null element of combination. So, the Semigroup Axiom A1 is valid.

Since D has greatest element \top , which is a support of all elements of Ψ_f , it follows that $\epsilon_\top(X) = X$, hence $\epsilon_\top([X]_\theta) = [\epsilon_\top(X)]_\theta = [X]_\theta$, so every element of Di_f/θ has \top as a support. Further, suppose that x is a support of $[X]_\theta$ such that $\epsilon_x(X) \equiv_\theta X$. Consider an element $y \in D$ such that $x \leq y$. We claim that then $\epsilon_y(X) \equiv_\theta X$. Indeed, if $\psi \in \epsilon_y(X)$, then $\psi = \epsilon_y(\phi) \leq \phi$ for some $\phi \in X$. On the other hand assume $\psi \in X$. By assumption, there is a ϕ in X such that $\psi \leq \epsilon_x(\phi)$. But $x \leq y$ implies $\epsilon_x(\phi) \leq \epsilon_y(\phi)$. So $\psi \leq \epsilon_y(\phi)$ for $\phi \in X$, and $\epsilon_y(\phi) \in \epsilon_y(X)$. This means that $\epsilon_y(X) \equiv_\theta X$. So y is also a support of $[X]_\theta$ and the Support Axiom A2 holds.

Note that $\{1\} \equiv_\theta \epsilon_x(\{1\})$, $\Psi_f \equiv_\theta \epsilon_x(\Psi_f)$. Below we show that the algebra is idempotent. In this case, from $[\epsilon_x(X)]_\theta = [\Psi_f]_\theta$ it follows $[X]_\theta = [\Psi_f]_\theta$ since $[\epsilon_x(X)]_\theta \leq [X]_\theta \leq [\Psi_f]_\theta$, so that the Unit and Null Axiom A3 is satisfied.

Next, assume $x \perp y | z$ and that x is a support of $[X]_\theta$, that is $\epsilon_x(X) \equiv_\theta X$. We want to show that this implies $\epsilon_y(X) \equiv_\theta \epsilon_y(\epsilon_z(X))$. In fact, consider an element $\psi \in \epsilon_y(\epsilon_z(X))$, that is $\psi = \epsilon_y(\epsilon_z(\phi))$ for some element ϕ from X . Then $\psi \leq \epsilon_y(\phi) \in \epsilon_y(X)$ since $\epsilon_z(\phi) \leq \phi$. On the other hand consider $\psi \in \epsilon_y(X)$. Then $\psi = \epsilon_y(\phi)$ for some $\phi \in X$ and from $\epsilon_x(X) \equiv_\theta X$ it follows that there is a $\chi \in X$ such that $\phi \leq \epsilon_x(\chi)$. Define $\eta = \epsilon_x(\chi)$ so that $\epsilon_x(\eta) = \eta$. Then we see that $\psi = \epsilon_y(\phi) \leq \epsilon_y(\eta) = \epsilon_y(\epsilon_z(\eta))$ by Axiom A4 in the information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. From this we obtain $\psi \leq \epsilon_y(\epsilon_z(\epsilon_x(\chi))) \leq \epsilon_y(\epsilon_z(\chi))$. Since $\chi \in X$, we have $\epsilon_y(\epsilon_z(\chi)) \in \epsilon_y(\epsilon_z(X))$. This shows that indeed $\epsilon_y(X) \equiv_\theta \epsilon_y(\epsilon_z(X))$, or also $\epsilon_y([X]_\theta) = \epsilon_y(\epsilon_z([X]_\theta))$. So the Extension Axiom A4 holds in the algebra $(Di_f, D; \leq, \perp, \cdot, \epsilon)$.

Further assume again $x \perp y | z$ and consider two elements $[X]_\theta$ and $[Y]_\theta$ from Di_f/θ having supports x and y respectively, that is $\epsilon_x(X) \equiv_\theta X$ and $\epsilon_y(Y) \equiv_\theta Y$. We claim that then $\epsilon_z(X \cdot Y) \equiv_\theta \epsilon_z(X) \cdot \epsilon_z(Y)$. Assume first that ψ belongs to $\epsilon_z(X) \cdot \epsilon_z(Y)$, which means that $\psi = \epsilon_z(\phi_1) \cdot \epsilon_z(\phi_2)$ for some elements $\phi_1 \in X$ and $\phi_2 \in Y$. But $\psi = \epsilon_z(\phi_1) \cdot \epsilon_z(\phi_2) \leq \epsilon_z(\phi_1 \cdot \phi_2)$ and the element $\epsilon_z(\phi_1 \cdot \phi_2)$ belongs to $\epsilon_z(X \cdot Y)$. On the other hand consider $\psi \in \epsilon_z(X \cdot Y)$, such that $\psi = \epsilon_z(\phi_1 \cdot \phi_2)$ for some elements $\phi_1 \in X$ and $\phi_2 \in Y$. From the assumptions that $\epsilon_x(X) \equiv_\theta X$ and $\epsilon_y(Y) \equiv_\theta Y$ it follows that there

are elements $\chi_1 \in X$ and $\chi_2 \in Y$ such that $\phi_1 \leq \epsilon_x(\chi_1)$ and $\phi_2 \leq \epsilon_x(\chi_2)$. Define $\eta_1 = \epsilon_x(\chi_1)$, $\eta_2 = \epsilon_y(\chi_2)$. These two element have support x and y respectively. Then we obtain $\psi = \epsilon_z(\phi_1 \cdot \phi_2) \leq \epsilon_z(\eta_1 \cdot \eta_2) = \epsilon_z(\eta_1) \cdot \epsilon_z(\eta_2)$ by Axiom A5 for the information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. From this it follows that $\psi \leq \epsilon_z(\epsilon_x(\chi_1)) \cdot \epsilon_z(\epsilon_y(\chi_2)) \leq \epsilon_z(\chi_1) \cdot \epsilon_z(\chi_2)$. The last element belongs to $\epsilon_z(X) \cdot \epsilon_z(Y)$. This proves that indeed $\epsilon_z(X \cdot Y) \equiv_\theta \epsilon_z(X) \cdot \epsilon_z(Y)$ or $\epsilon_z([X]_\theta \cdot [Y]_\theta) = \epsilon_z([X]_\theta) \cdot \epsilon_z([Y]_\theta)$. So, the Combination Axion A5 holds too.

Finally, to verify Idempotency, A6, we show that $\epsilon_x(X) \cdot X \equiv_\theta X$. If $\psi \in X$, then by the idempotency in the original information algebra $\psi = \epsilon_x(\psi) \cdot \psi$ and this element belongs to $\epsilon_x(X) \cdot X$. Conversely if $\psi \in \epsilon_x(X) \cdot X$, then $\psi = \epsilon_x(\phi_1) \cdot \phi_2$ and ϕ_1 and ϕ_2 belong to X . Since X is directed, there is a $\phi \in X$ which dominates ϕ_1 and ϕ_2 , such that $\psi \leq \phi \in X$. This shows that $\epsilon_x(X) \cdot X \equiv_\theta X$, hence $\epsilon_x([X]_\theta) \cdot [X]_\theta = [X]_\theta$. Therefore, the algebra $(Di_f, D; \leq, \perp, \cdot, \epsilon)$ is idempotent

So far we have shown that $(Di_f/\theta, D; \leq, \perp, \cdot, \epsilon)$ is an idempotent generalised information algebra. It remains to show that this algebra is algebraic. The proof will be based on Theorem 7.3. As a preparation we prove the following lemma.

Lemma 7.2 *The relation $[X]_\theta \leq [Y]_\theta$ holds if and only if for all $\phi \in X$ there is a $\psi \in Y$ such that $\phi \leq \psi$.*

Proof. The relation $[X]_\theta \leq [Y]_\theta$ means that $[X]_\theta \cdot [Y]_\theta = [X \cdot Y]_\theta = [Y]_\theta$ or $X \cdot Y \equiv_\theta Y$. So, assume $X \cdot Y$ is equivalent to Y . Consider an element $\phi \in X$. Then for any element $\phi \cdot \chi$ in $X \cdot Y$, where $\chi \in Y$, there is an element $\psi \in Y$ so that $\phi \cdot \chi \leq \psi$. But then $\phi \leq \psi$.

If, on the contrary for any $\phi \in X$ there is a $\psi \in Y$ such that $\phi \leq \psi$, then $\phi \cdot \psi = \psi$ and $\phi \cdot \psi \in X \cdot Y$. And if $\psi \in Y$, then $\psi \leq \psi \cdot \phi$ for any $\phi \in X$. so, indeed $X \cdot Y \equiv_\theta Y$. \square

Now, we resume the proof of the theorem. We take the set of classes $[\{\psi\}]_\theta$ for $\psi \in \Psi_f$ to be the set of Ψ' of Theorem 7.3. This set will turn out to be the set of finite elements of the algebra $(Di_f/\theta, D; \leq, \perp, \cdot, \epsilon)$.

Let X be a directed subset of Ψ' and define $X' = \{\psi : [\{\psi\}]_\theta \in X\}$. To simplify notation, we write subsequently $[\psi]_\theta$ instead of $[\{\psi\}]_\theta$. The set X' is directed in Ψ_f . By Lemma 7.2, $[\psi]_\theta \leq [X']_\theta$ if $\psi \in X'$. So, $[X']_\theta$ is an upper bound of X . Assume that $[Y]_\theta$ is another upper bound of X . Then, for all $[\psi]_\theta \in X$, there is a $\chi \in Y$ such that $\psi \leq \chi$. Therefore (Lemma 7.2) $[X']_\theta \leq [Y]_\theta$ and $[X']_\theta$ is the supremum of X , that is $[X']_\theta = \vee X$. So item 1 of Theorem 7.3 holds.

Next, consider an element $[X]_\theta$ with support x in Di_f/θ , that is $[X]_\theta = \epsilon_x([X]_\theta) = [\epsilon_x(X)]_\theta$. We claim that

$$\epsilon_x(X) \equiv_\theta \{\psi \in \Psi_f : \psi = \epsilon_x(\psi) \leq \phi \text{ for some } \phi \in X\}. \quad (7.10)$$

In fact, assume $\psi \in \epsilon_x(X)$, such that $\psi = \epsilon_x(\phi)$ for some $\phi \in X$. Then $\psi \in \Psi_f$, (remember that we assume that Ψ_f is closed under extraction), and $\psi = \epsilon_x(\psi) \leq \phi \in X$. On the other hand $\psi = \epsilon_x(\psi) \leq \phi \in X$ implies $\psi \leq \epsilon_x(\phi)$ and $\epsilon_x(\phi) \in \epsilon_x(X)$. This proves that (7.10) holds. So, we conclude, using the same argument as above, that

$$\begin{aligned} \epsilon_x([X]_\theta) &= [\{\psi \in \Psi_f : \psi = \epsilon_x(\psi) \leq \phi \text{ for some } \phi \in X\}]_\theta \\ &= \bigvee \{[\psi]_\theta \in \Psi' : [\psi]_\theta = \epsilon_x([\psi]_\theta) \leq [X]_\theta\}. \end{aligned} \quad (7.11)$$

This verifies item 2, density, of Theorem 7.3.

Finally, assume $[\psi]_\theta \leq \bigvee X$, $\psi \in \Psi_f$, for some directed subset X of Ψ' . Define $X' = \{\phi : [\phi]_\theta \in X\}$. This set is also directed. So, as above, $[\psi]_\theta \leq \bigvee X = [X']_\theta$. Then, by Lemma 7.2 there is an element $\phi \in X'$ such that $\psi \leq \phi$, hence $[\psi]_\theta \leq [\phi]_\theta \in X$. This proves item 3 of Theorem 7.3. So, the information algebra $(Di_f/\theta, D; \leq, \perp, \cdot, \epsilon)$ is according to Theorem 7.3 algebraic and the elements $[\psi]_\theta$ for $\psi \in \Psi_f$ are its finite elements.

It remains to show that the map $\phi \mapsto [\{\psi \in \Psi_f : \psi \leq \phi\}]_\theta$ is an embedding. Define $A_\phi = \{\psi \in \Psi_f : \psi \leq \phi\}$. First we note that the map $\phi \mapsto [A_\phi]_\theta$ is one-to-one. In fact, if $[A_\phi]_\theta = [A_\psi]_\theta$, or $A_\phi \equiv_\theta A_\psi$, then it follows from density in the compact information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ that $\phi = \bigvee A_\phi = \bigvee A_\psi = \psi$.

Next we verify that the map is a homomorphism. In order to show that $\phi \cdot \psi \mapsto [A_{\phi \cdot \psi}]_\theta = [A_\phi \cdot A_\psi]_\theta = [A_\phi]_\theta \cdot [A_\psi]_\theta$, it is sufficient to prove that $\bigvee A_{\phi \cdot \psi} = \bigvee (A_\phi \cdot A_\psi)$, since the first supremum exists in a compact information algebra, that is, to prove that

$$\bigvee \{\phi' \cdot \psi' : \phi', \psi' \in \Psi_f, \phi' \leq \phi, \psi' \leq \psi\} = \bigvee \{\psi' \in \Psi_f : \psi' \leq \phi \cdot \psi\} = \phi \cdot \psi.$$

Note that by density, the second supremum exists in Ψ and it is an upper bound of the set on the left hand side. If η is another upper bound of this set, then $\phi', \psi' \leq \eta$. Since by density both ϕ and ψ are the suprema of the finite ϕ', ψ' they dominate, we conclude that $\phi, \psi \leq \eta$, hence $\phi \cdot \psi \leq \eta$ and $\phi \cdot \psi$ is indeed the supremum of the set on the left hand side. So, we have proved that $\phi \cdot \psi \mapsto [A_\phi]_\theta \cdot [A_\psi]_\theta$. In addition, $1 \mapsto [1]_\theta$ and $0 \mapsto [\Psi_f]_\theta$, the unit and null elements in $(Di_f/\theta, D; \leq, \perp, \cdot, \epsilon)$.

Finally, we must show that $\epsilon_x(\phi) \mapsto [A_{\epsilon_x(\phi)}]_\theta = \epsilon_x([A_\phi]_\theta) = [\epsilon_x(A_\phi)]_\theta$. By density in the algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$,

$$\begin{aligned} \epsilon_x(\phi) &= \bigvee \{\psi \in \Psi_f : \psi \leq \epsilon_x(\phi)\} \\ &= \bigvee \{\psi \in \Psi_f : \psi = \epsilon_x(\psi) \leq \phi\} \\ &= \bigvee \{\epsilon_x(\psi) : \psi \in \Psi_f : \psi \leq \phi\} \\ &= \bigvee \epsilon_x(A_\phi). \end{aligned} \quad (7.12)$$

So, we have $\bigvee A_{\epsilon_x(\phi)} = \bigvee \epsilon_x(A_\phi)$ which implies $[A_{\epsilon_x(\phi)}]_\theta = \epsilon_x([A_\phi]_\theta)$.

Thus, the map $\phi \mapsto [A_\phi]_\theta$ is an embedding. This concludes the proof. \square

The embedding $\phi \mapsto [A_\phi]_\theta$ is not only an ordinary information algebra homomorphism. It is in fact a *continuous* map, in the order-theoretic sense.

Theorem 7.7 *Assume $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ to be a domain-free compact information algebra. Then, if X is a directed subset of Ψ with a supremum in Ψ ,*

$$[A_{\bigvee X}]_\theta = \bigvee_{\phi \in X} [A_\phi]_\theta. \quad (7.13)$$

Proof. Since $\phi \in X$ implies $\phi \leq \bigvee X$, we have $[A_\phi]_\theta \leq [A_{\bigvee X}]_\theta$. So, $[A_{\bigvee X}]_\theta$ is an upper bound for the $[A_\phi]_\theta$ with $\phi \in X$. If $[Y]$ is another upper bound of this set, then consider an element $\psi \in A_{\bigvee X}$, such that $\psi \in \Psi_f$ and $\psi \leq \bigvee X$. Then, by compactness in the algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$, there is a $\phi \in X$ such that $\psi \leq \phi$, hence $\psi \in A_\phi$. From $[A_\phi]_\theta \leq [Y]_\theta$ it follows that there is $\chi \in Y$ such that $\psi \leq \chi$. But this shows that $[A_{\bigvee X}]_\theta \leq [Y]_\theta$ (Lemma 7.2). So, $[A_{\bigvee X}]_\theta$ is the supremum of the $[A_\phi]_\theta$ for $\phi \in X$. \square

We could also have considered the ideal completion of the finite elements of the compact information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. By Theorem 7.4 we would obtain an algebraic information algebra. In fact, this algebra is isomorphic to the algebra $(\text{Dif}/\theta, D; \leq, \perp, \cdot, \epsilon)$, if the finite elements are closed under extraction, and the isomorphism is a continuous map.

Theorem 7.8 *Assume $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ to be a domain-free compact information algebra, where $(D; \leq)$ has a greatest element \top and the set of finite elements is closed under extraction. Then the two algebraic information algebras $(\text{Dif}/\theta, D; \leq, \perp, \cdot, \epsilon)$ and $(I_{\Psi_f}, D; \leq, \perp, \cdot, \epsilon)$ are isomorphic under a continuous map.*

Proof. We show first that any directed set X in Ψ_f is equivalent to the ideal $I(X)$ it generates in Ψ_f , that is $X \equiv_\theta I(X)$. If $\phi \in X$, then $\phi \leq \phi$ and $\phi \in I(X)$. Conversely, if $\psi \in I(X)$, then there is a finite set of elements ψ_1, \dots, ψ_n in X such that $\psi \leq \psi_1 \vee \dots \vee \psi_n$ (see (7.3)). Since X is directed, there is an element $\phi \in X$ which dominates all ψ_i , $i = 1, \dots, n$, hence $\psi \leq \phi$. So, indeed $X \equiv_\theta I(X)$. This means that any equivalence class $[X]_\theta$ in Dif/θ can be represented by the ideal $I(X)$, that is $[X]_\theta = [I(X)]_\theta$.

Consider now the map $[X]_\theta \mapsto I(X)$ from Dif/θ into I_{Ψ_f} . By the last remark, this map is well defined. It is onto I_{Ψ_f} , since any ideal I in Ψ_f is directed and $[I]_\theta \mapsto I(I) = I$. It is also one-to-one: Suppose that $I(X) = I(Y)$, then $X \equiv_\theta I(X)$ and $Y \equiv_\theta I(Y)$, therefore $X \equiv_\theta Y$ or $[X]_\theta = [Y]_\theta$.

Next, we verify that the map is a homomorphism. First we show that $[X]_\theta \cdot [Y]_\theta$ maps to $I(X) \cdot I(Y)$, that is, $I(X \cdot Y) = I(X) \cdot I(Y)$. Consider an element $\phi \in I(X \cdot Y)$. Then there is a finite set of elements ψ_1, \dots, ψ_n in $X \cdot Y$ such that $\phi \leq \psi_1 \cdot \dots \cdot \psi_n$ and each element ψ_i equals $\psi_{i,1} \cdot \psi_{i,2}$ for some $\psi_{i,1} \in X$ and $\psi_{i,2} \in Y$. Since both X and Y are directed, there are elements $\psi_1 \in X$ and $\psi_2 \in Y$ which dominate $\psi_{1,1}, \dots, \psi_{n,1}$ and $\psi_{1,2}, \dots, \psi_{n,2}$ respectively. Then it follows that $\phi \leq \psi_1 \cdot \psi_2$, which shows that $\phi \in I(X) \cdot I(Y)$.

Conversely, assume $\phi \in I(X) \cdot I(Y)$, which means that $\phi \leq \psi_1 \cdot \psi_2$ for some elements $\psi_1 \in I(X)$ and $\psi_2 \in I(Y)$. Then $\psi_1 \leq \psi_{1,1} \cdot \dots \cdot \psi_{n,1}$ for some elements $\psi_{1,1}, \dots, \psi_{n,1}$ of X and similarly $\psi_2 \leq \psi_{1,2} \cdot \dots \cdot \psi_{m,2}$ for some elements $\psi_{1,2}, \dots, \psi_{m,2}$ of Y . The sets X and Y are directed, therefore there are elements $\phi_1 \in X$ and $\phi_2 \in Y$, which dominate $\psi_{1,1}, \dots, \psi_{n,1}$ and $\psi_{1,2}, \dots, \psi_{m,2}$ respectively, hence $\phi \leq \phi_1 \cdot \phi_2 \in X \cdot Y$. This shows that $\phi \in I(X \cdot Y)$ and therefore $I(X \cdot Y) = I(X) \cdot I(Y)$.

Further, $[\{1\}]_\theta$ maps to $I(\{1\}) = \{1\}$, and $[\Psi_f]_\theta$ maps to Ψ_f .

To complete the verification that the map is a homomorphism, we show that $\epsilon_x([X]_\theta) = [\epsilon_x(X)]_\theta$ maps to $\epsilon_x(I(X))$ by proving that $I(\epsilon_x(X)) = \epsilon_x(I(X))$. Let first $\phi \in I(\epsilon_x(X))$ such that $\phi \leq \psi_1 \cdot \dots \cdot \psi_n$ for some elements ψ_1, \dots, ψ_n of $\epsilon_x(X)$. This means that $\psi_i = \epsilon_x(\phi_i)$ for some $\phi_i \in X$. As X is directed, there is a $\chi \in X$ such that $\phi_1, \dots, \phi_n \leq \chi$. It follows then that $\phi \leq \epsilon_x(\phi_1) \cdot \dots \cdot \epsilon_x(\phi_n) \leq \epsilon_x(\phi_1 \cdot \dots \cdot \phi_n) \leq \epsilon_x(\chi)$. This shows that $\phi \in \epsilon_x(I(X))$.

If, on the other hand, $\phi \in \epsilon_x(I(X))$, then there is an element $\psi \in I(X)$ such that $\phi \leq \epsilon_x(\psi)$ and further $\psi \leq \psi_1 \cdot \dots \cdot \psi_n$ for elements $\psi_i \in X$. Then there is an element $\chi \in X$, such that $\psi_1, \dots, \psi_n \leq \chi$. This implies $\phi \leq \epsilon_x(\psi) \leq \epsilon_x(\psi_1 \cdot \dots \cdot \psi_n) \leq \epsilon_x(\chi) \in \epsilon_x(X)$. Therefore we have $\phi \in I(\epsilon_x(X))$, hence $I(\epsilon_x(X)) = \epsilon_x(I(X))$.

So the map $[X]_\theta \mapsto I(X)$ is an information algebra isomorphism. Finally, we show that the map is continuous. A map f from an algebraic lattice $(\Phi; \leq)$ to another algebraic lattice is continuous, if (Davey & Priestley, 2002)

$$f(\phi) = \bigvee \{f(\psi) : \psi \in \Phi_f, \psi \leq \phi\}.$$

In our case, $f([X]_\theta) = I(X)$ and the finite elements are the classes $[\psi]_\theta$ for $\psi \in \Psi_f$. So we must verify that

$$I(X) = \bigvee \{I(\{\psi\}) : \psi \in \Psi_f, [\psi]_\theta \leq [X]_\theta\}. \quad (7.14)$$

Now, $[\psi]_\theta \leq [X]_\theta$ if and only if there is a $\phi \in X$ such that $\psi \leq \phi$. Therefore, we need to verify that

$$I(X) = \bigvee \{I(\{\psi\}) : \psi \in \Psi_f, \psi \leq \phi \text{ for some } \phi \in X\}.$$

We may identify the principal ideal $I(\{\psi\})$ with ψ by the embedding of the finite elements of Ψ into their ideal completion (see Section 7.1). Then the

last equality becomes

$$I(X) = \bigvee \{\psi : \psi \in \Psi_f, \psi \leq \phi \text{ for some } \phi \in X\} \leq \vee X.$$

But $I(X) = \bigvee X$ in the ideal completion. This proves (7.14). \square

In view of these results, we have two equivalent ways to complete a compact information algebra to an algebraic one. One method is by ideal completion, the other one by adjoining the missing suprema.

We remark that compact and algebraic generalised information algebras may be obtained from algebraic semirings; this has been shown for idempotent algebraic valuation algebras in (Guang & Kohlas, 2015).

Here is a simple example of an algebraic information algebra:

Example 7.1 *Information Algebra of Strings:* Consider a finite alphabet Σ , the set Σ^* of finite strings over Σ , including the empty string ϵ , and the set Σ^ω of infinite strings over Σ . Let $\Sigma^{**} = \Sigma^* \cup \Sigma^\omega \cup \{0\}$, where 0 is a symbol not contained in Σ . For two strings $r, s \in \Sigma^{**}$, define $r \leq s$, if r is a prefix of s or if $s = 0$. The empty string is a prefix of any string. Define a combination operation in Σ^{**} as follows:

$$r \cdot s = \begin{cases} s, & \text{if } r \leq s, \\ r & \text{if } s \leq r, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, (Σ^{**}, \cdot) is a commutative idempotent semigroup. The empty string ϵ is the unit element, and the adjoined element 0 is the null element of combination. For extraction, we define operators ϵ_n for any $n \in \mathbb{N}$ and also for $n = \infty$. Let $\epsilon_n(s)$ be the prefix of length n of string s , if the length of s is at least n , and let $\epsilon_n(s) = s$ otherwise. In particular, define $\epsilon_\infty(s) = s$ for any string s and $\epsilon_n(0) = 0$ for any n . It is easy to verify that the axioms of an idempotent valuation algebra are satisfied. This is the so-called *string algebra* $(\Sigma^{**}, E; \cdot, \epsilon, 0, \circ)$. Obviously, the finite strings are the finite elements of this algebra and any infinite string is the supremum of the its finite prefixes. For finite n the strings with $\epsilon_n(s) = s$ are finite of length less than n , hence all finite so that density holds trivially. For infinite n density holds since for all finite strings we have $\epsilon_\infty(s) = s$. So, the information algebra of strings is *compact*. In fact, Σ^{**} is clearly a dcpo, so the algebra is even *algebraic*. \ominus

7.3 Duality For Compact Algebras

In this section we turn back to duality between domain-free and labeled generalised information algebras. What is a compact or algebraic labeled information algebras? This question will be examined by looking at the

labeled version of a compact or algebraic domain-free information algebra in order to see how compactness transforms into the labeled version. Then, based on this analysis, we study how duality extends to compact and algebraic information algebras.

Consider a compact domain-free generalised information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. According to the previous discussions, we assume that there is a greatest domain \top in D . As we have seen, we can always adjoin such a domain, if necessary. So there is no loss of generality. We form the dual labeled algebra $(\Phi, D; \leq, \perp, \cdot, t)$, where Φ is the set of pairs (ϕ, x) with $\phi \in \Psi$ and $\epsilon_x(\phi) = \phi$, see Section 5.3. In particular, let Φ_x be the set of all pairs (ϕ, x) for a fixed x . Then

$$\Phi = \bigcup_{x \in D} \Phi_x.$$

Note that idempotency allows, as in the domain-free case, to define a partial order in Φ . In fact, define $(\phi, x) \leq (\psi, y)$ if and only if $(\phi, x) \cdot (\psi, y) = (\phi \cdot \psi, x \vee y) = (\psi, y)$. This implies $\phi \cdot \psi = \psi$ or $\phi \leq \psi$ in (Ψ, \leq) and $x \leq y$ in $(D; \leq)$.

As a preparation, we prove two useful results about the labeled algebra $(\Phi, D; \leq, \perp, \cdot, t)$.

Lemma 7.3 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be an idempotent domain-free generalised information algebra and $(\Phi, D; \leq, \perp, \cdot, t)$ its dual labeled version. If the supremum of a subset X of Φ exists in Φ , then*

$$\bigvee X = (\bigvee_{(\phi, x) \in X} \phi, \bigvee_{(\phi, x) \in X} x). \quad (7.15)$$

Proof. Assume $\bigvee X = (\chi, y)$. Then $(\phi, x) \leq (\chi, y)$ for all $(\phi, x) \in X$, hence $\phi \leq \chi$ and $x \leq y$. Consider other upper bounds χ' and y' for the elements ϕ and x , $(\phi, x) \in X$. Then $(\phi, x) \leq (\phi', y')$, hence $(\chi, y) \leq (\chi', y')$. But this implies $\chi \leq \chi'$ and $y \leq y'$ and so indeed $\chi = \bigvee_{(\phi, x) \in X} \phi$ and $y = \bigvee_{(\phi, x) \in X} x$. This is (7.15). \square

Lemma 7.4 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be an idempotent domain-free generalised information algebra and $(\Phi, D; \leq, \perp, \cdot, t)$ its dual labeled version. Let X be a subset of Ψ such that $\epsilon_x(X) = X$, that is, all elements of X have support x . If the supremum of X exists in Ψ , then $(\bigvee X, x) \in \Phi$ and*

$$\bigvee_{\psi \in X} (\psi, x) = (\bigvee X, x).$$

Proof. We need only to show that $\bigvee X$ has support x . Define $\phi = \bigvee X$. Then, for all $\psi \in X$ we have $\psi = \epsilon_x(\psi) \leq \phi$, hence $\psi = \epsilon_x(\psi) \leq \epsilon_x(\phi)$. So, $\epsilon_x(\phi)$ is an upper bound of X , therefore $\phi \leq \epsilon_x(\phi)$, hence $\phi = \epsilon_x(\phi)$. \square

The next theorem shows how finite elements in $(\Phi_x; \leq)$ relate to finite elements in $(\Psi; \leq)$.

Theorem 7.9 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be a domain-free compact information algebra with finite elements Ψ_f and $(\Phi, D; \leq, \perp, \cdot, t)$ its dual labeled version. Then $(\psi, x) \in \Phi$ is finite in $(\Phi_x; \leq)$ if and only if ψ is finite in $(\Psi; \leq)$, that is, $\psi \in \Psi_f$.*

Proof. Consider an element (ψ, x) of Φ with $\psi \in \Psi_f$. Let X be a directed subset of Φ_x whose supremum exists in $(\Phi_x; \leq)$ and such that $(\psi, x) \leq \vee X$. Define $X' = \{\phi \in \Psi : (\phi, x) \in X\}$. Clearly, X' is directed too and since $\vee X = (\vee X', x)$ (Lemma 7.3) the supremum of X' exists in Ψ and $\psi \leq \vee X'$. Since ψ is finite in $(\Psi; \leq)$ there is a $\phi \in X'$ such that $\psi \leq \phi$, hence $(\psi, x) \leq (\phi, x) \in X$. This shows that (ψ, x) is finite in $(\Phi_x; \leq)$.

Conversely, assume that (ψ, x) is finite in $(\Phi_x; \leq)$. Let X be a directed subset of Ψ whose supremum exists in Ψ and such that $\psi \leq \vee X$. Then we have $\psi = \epsilon_x(\psi) \leq \epsilon_x(\vee X) = \vee \epsilon_x(X)$ (Theorem 7.2). Define $X' = \{(\epsilon_x(\phi), x) : \phi \in X\}$. It is a directed set in $(\Phi_x; \leq)$ and we have $(\psi, x) \leq (\vee \epsilon_x(X), x) = \vee X'$ (Lemma 7.4). Since (ψ, x) is assumed to be finite in $(\Phi_x; \leq)$ there is an element $(\epsilon_x(\phi), x) \in X'$ such that $(\psi, x) \leq (\epsilon_x(\phi), x)$. This implies $\psi \leq \phi$ for an element $\phi \in X$. This shows that ψ is finite in $(\Psi; \leq)$. \square

According to this theorem, finite elements in $(\Psi; \leq)$ correspond to finite elements in $(\Phi_x; \leq)$ for domains x which are supports of the finite elements in $(\Psi; \leq)$. Note that finite elements in $(\Phi_x; \leq)$ are not necessarily finite in $(\Phi; \leq)$ and that the finite elements in $(\Psi; \leq)$ do not induce finite elements in $(\Phi; \leq)$, as one might have expected. So, if we denote the finite elements in $(\Phi_x; \leq)$ by $\Phi_{x,f}$, and

$$\Phi_f = \bigcup_{x \in D} \Phi_{x,f},$$

then Φ_f does not represent the finite elements of $(\Phi; \leq)$ but the union of the locally finite ones. Note that if $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is a compact information algebra, then Φ_f is closed under combination. In fact, if $(\phi, x) \in \Phi_{x,f}$ and $(\psi, y) \in \Phi_{y,f}$, then by Theorem 7.9 ϕ and ψ are finite elements in $(\Psi; \leq)$ and so is its combination $\phi \cdot \psi$. This combination has $x \vee y$ as a support and again by the same theorem, therefore $(\phi, x) \cdot (\psi, y) = (\phi \cdot \psi, x \vee y)$ are finite in $\Phi_{x \vee y, f}$. However, transport of finite elements keeps them not necessarily finite, except if the finite elements of $(\Psi; \leq)$ are closed under extraction. Nevertheless, for $x \leq y$, the element $t_y(\psi, x) = (\psi, x) \cdot (1, y)$ remains finite, if (ψ, x) is finite. This is true because $(1, y)$ is a finite element.

Next we show that strong density of the compact algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ induces local density within the domains Φ_x of the dual labeled algebra. That is, the finite elements in $(\Phi_x; \leq)$ are dense in Φ_x and approximate thus the elements of Φ_x .

Theorem 7.10 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be a domain-free compact information algebra and $(\Phi, D; \leq, \perp, \cdot, t)$ its dual labeled version. Then, for all $(\phi, x) \in \Phi$,*

$$(\phi, x) = \bigvee \{(\psi, x) \in \Phi_{x,f} : (\psi, x) \leq (\phi, x)\}. \quad (7.16)$$

Proof. By strong density in the algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ we have

$$\begin{aligned} (\phi, x) &= \left(\bigvee \{ \psi \in \Psi_f : \psi = \epsilon_x(\psi) \leq \phi \}, x \right) \\ &= \bigvee \{ (\psi, x) \in \Phi_{x,f} : (\psi, x) \leq (\phi, x) \}. \end{aligned}$$

This equality holds by Lemma 7.4. \square

So, the dual, labeled version of a compact information algebra is a labeled algebra, where local density according to (7.16) holds. We take this as the model to define labeled compact information algebras. Note that order in a labeled information algebra $(\Phi, D; \leq, \perp, \cdot, t)$ is defined again by $\phi \leq \psi$ if $\phi \cdot \psi = \psi$. This induces also a partial order in $(\Phi_x; \leq)$ between the elements $\Phi_x = \{\phi \in \Phi : d(\phi) = x\}$ in domain x . The following lemma states a few elementary properties of this labeled order.

Lemma 7.5 *Let $(\Phi, D; \leq, \perp, \cdot, t)$ be an idempotent labeled generalised information algebra. Then*

1. $x \leq d(\phi)$ implies $t_x(\phi) \leq \phi$,
2. $x \geq d(\phi)$ implies $t_x(\phi) \geq \phi$,
3. $\phi \leq \psi$ implies $t_x(\phi) \leq t_x(\psi)$ for any $x \in D$,
4. $\phi, \psi \leq \phi \cdot \psi$,
5. $\phi \leq \psi$ implies $\phi \cdot \chi \leq \psi \cdot \chi$ for any $\chi \in \Phi$.

Proof. 1.) follows from the Idempotency Axiom A7 of a labeled generalised information algebra, $t_x(\phi) \cdot \phi = \phi$.

2.) follows from $t_x(\phi) = \phi \cdot 1_x$, hence by idempotency, $t_x(\phi) \cdot \phi = \phi \cdot 1_x \cdot \phi = \phi \cdot 1_x = t_x(\phi)$.

3.) Let $d(\phi) = y$ and $d(\psi) = z$ and assume first $x \geq y, z$. Then $y \perp x | x$, hence $y \perp z | x$. Using the Combination Axiom A5, it follows that $t_x(\psi) = t_x(\phi \cdot \psi) = t_x(\phi) \cdot t_x(\psi)$, which shows that $t_x(\phi) \leq t_x(\psi)$ in this case. Assume next, that $d(\phi) = d(\psi) = y$ and $x \leq y$. By item 1 proved above, we have $t_x(\phi) \cdot \phi = \phi$, hence, if $\phi \leq \psi$, we obtain $t_x(\phi) \cdot \psi = \psi$. From $y \perp x | x$ it follows with the Combination Axiom that $t_x(\psi) = t_x(t_x(\phi) \cdot \psi) = t_x(\phi) \cdot t_x(\psi)$ which shows that in this case too $t_x(\phi) \leq t_x(\psi)$. In the general case with $d(\phi) = y$ and $d(\psi) = z$, the assumption $\phi \leq \psi$ implies first $t_{x \vee y \vee z}(\phi) \leq t_{x \vee y \vee z}(\psi)$ and then $t_x(\phi) = t_x(t_{x \vee y \vee z}(\phi)) \leq t_x(t_{x \vee y \vee z}(\psi)) = t_x(\psi)$, using the two special cases proved before.

- 4.) follows from idempotency, $\phi \cdot (\phi \cdot \psi) = \phi \cdot \psi$ and $\psi \cdot (\phi \cdot \psi) = \phi \cdot \psi$.
 5.) If $\phi \leq \psi$, we have by idempotency $(\phi \cdot \chi) \cdot (\psi \cdot \chi) = (\phi \cdot \psi) \cdot \chi = \psi \cdot \chi$. \square

The lemma shows in particular, that the combination and the transport operations preserve order.

We remark that we may always adjoin a greatest element \top to $(D; \leq)$ in a generalised domain-free information algebra, as was explained in Section 7.1. This holds also for compact domain-free information algebras, since (weak) density guarantees local density in the domain \top . So, we may assume without loss of generality that the derived labeled information algebra has also a greatest domain.

After this preparation, we are in a position to define the concept of a labeled compact information algebra.

Definition 7.4 *An idempotent labeled generalised information algebra $(\Phi, D; \leq, \perp, \cdot, t)$ is called compact, if $(D; \leq)$ has a greatest element \top , and*

1. *for all domains $x \in D$ and elements ϕ with $d(\phi) = x$,*

$$\phi = \bigvee \{\psi \in \Phi_{x,f} : \psi \leq \phi\}, \quad (7.17)$$

where $\Phi_{x,f}$ denotes the set of the finite elements of $(\Phi_x; \leq)$.

2. *If $\phi \in \Phi_{x,f}$ and $y \geq x$, then $t_y(\phi) \in \Phi_{y,f}$.*

Let

$$\Phi_f = \bigcup_{x \in D} \Phi_{x,f}$$

be the set of all locally finite elements. Again, we emphasise that this is not the set of the finite elements of $(\Phi; \leq)$.

The justification of this definition of compact labeled information algebras will be that the associated dual domain-free information $(\Phi/\sigma, D; \leq, \perp, \cdot, \epsilon)$ is again compact. Before we show this, we give some useful results.

Lemma 7.6 *Let $(\Phi, D; \leq, \perp, \cdot, t)$ be a labeled compact information algebra, X a directed subset of Φ_y such that its supremum exists in Φ_y and $x \leq y$. Then*

$$t_x(\bigvee X) = \bigvee t_x(X). \quad (7.18)$$

Proof. Assume first $\phi \in X$ such that $\phi \leq \bigvee X$, hence $t_x(\phi) \leq t_x(\bigvee X)$. So, $t_x(\bigvee X)$ is an upper bound of the elements $t_x(\phi)$ for $\phi \in X$.

On the other hand, by density in the compact labeled algebra,

$$\begin{aligned} t_x(\bigvee X) &= \bigvee \{\psi \in \Phi_{x,f} : \psi \leq t_x(\bigvee X)\} \\ &= \bigvee \{\psi \in \Phi_{x,f} : t_y(\psi) \leq \bigvee X\}. \end{aligned} \quad (7.19)$$

Since $t_y(\psi)$ is finite in domain y , if ψ is so in domain $x \leq y$, there is an element $\phi \in X$ such that $t_y(\psi) \leq \phi$ if $t_y(\psi) \leq \bigvee X$. But then it follows that $\psi \leq t_x(\phi) \in t_x(X)$ and therefore $t_x(\bigvee X)$ is the least upper bound of $t_x(X)$. So, indeed $t_x(\bigvee X) = \bigvee t_x(X)$. \square

This lemma implies that Φ_f is closed under combination. In fact, consider $\phi \in \Phi_{x,f}$ and $\psi \in \Phi_{y,f}$, and a directed set X in $\Phi_{x \vee y}$ such that $\phi \cdot \psi \leq \bigvee X$. Then $\phi \leq t_x(\bigvee X) = \bigvee t_x(X)$ by Lemma 7.6 and similarly $\psi \leq t_y(\bigvee X) = \bigvee t_y(X)$. Both sets $t_x(X)$ and $t_y(X)$ are directed, and therefore there are elements $t_x(\phi') \in t_x(X)$ such that $\phi \leq t_x(\phi')$ and $t_y(\psi') \in t_y(X)$ such that $\psi \leq t_y(\psi')$. Both ϕ', ψ' belong to X and so there is also an element χ in X such that $\phi', \psi' \leq \chi$. Hence, finally we conclude that $\phi \cdot \psi \leq \phi' \cdot \psi' \leq \chi \in X$. This proves that $\phi \cdot \psi \in \Phi_{x \vee y, f}$, hence $\phi \cdot \psi$ belongs to Φ_f . But Φ_f is not necessarily closed under transport.

As a preparation for the examination of the dual domain-free algebra associated with a labeled compact information algebra $(\Phi, D; \leq, \perp, \cdot, t)$ we prove the following lemma. We recall that the congruence \equiv_σ is defined in Section 5.1.

Lemma 7.7 *Let $(\Phi, D; \leq, \perp, \cdot, t)$ be an idempotent labeled information algebra, X a directed subset of Φ such that its supremum exists in Φ . Then in $(\Phi/\sigma, D; \leq, \perp, \cdot, \epsilon)$,*

$$[\bigvee X]_\sigma = \bigvee [X]_\sigma, \quad (7.20)$$

where $[X]_\sigma = \{[\phi]_\sigma : \phi \in X\}$.

Proof. Define $\psi = \bigvee X$ such that $[\psi]_\sigma = [\bigvee X]_\sigma$ and assume that $d(\psi) = x$. Then, for all $\phi \in X$ we have $\phi \leq \psi$ and $d(\phi) \leq x$. Therefore, for all $\phi \in X$ we have $[\phi]_\sigma \leq [\psi]_\sigma$ and so $[\psi]_\sigma$ is an upper bound of $[X]_\sigma$.

Assume $[\chi]_\sigma$ to be another upper bound of $[X]_\sigma$ and $d(\chi) = y$. For any ϕ in X we have $[\chi]_\sigma = [\phi]_\sigma \cdot [\chi]_\sigma = [\phi \cdot \chi]_\sigma = [t_{x \vee y}(\phi) \cdot t_{x \vee y}(\chi)]_\sigma$. This implies $t_{x \vee y}(\phi) \leq t_{x \vee y}(\chi)$. Since for $\phi \in X$ we have $d(\phi) \leq x$, it follows that $\phi \leq t_x(\phi) = t_x(t_{x \vee y}(\phi)) \leq t_x(t_{x \vee y}(\chi))$. But then $\psi = \bigvee X \leq t_x(t_{x \vee y}(\chi))$. It follows that $t_{x \vee y}(\psi) \leq t_{x \vee y}(t_x(t_{x \vee y}(\chi))) \leq t_{x \vee y}(t_{x \vee y}(\chi)) = t_{x \vee y}(\chi)$. From this we conclude that $[\psi]_\sigma \leq [\chi]_\sigma$, such that $[\psi]_\sigma$ is the supremum of $[X]_\sigma$. \square

Now we show that the domain-free information algebra $(\Phi/\sigma, D; \leq, \perp, \cdot, \epsilon)$ associated with a labeled compact information algebra $(\Phi, D; \leq, \perp, \cdot, t)$ is indeed again compact. This justifies the definition of a labeled compact information algebra above.

Theorem 7.11 *Let $(\Phi, D; \leq, \perp, \cdot, t)$ be a labeled compact information algebra. Then $(\Phi/\sigma, D; \leq, \perp, \cdot, \epsilon)$ is a domain-free compact information algebra and its finite elements are the elements $[\psi]_\sigma$ for $\psi \in \Phi_f$.*

Proof. We know already that $(\Phi/\sigma, D; \leq, \perp, \cdot, \epsilon)$ is an idempotent domain-free information algebra (see Section 5.1, in particular Theorem 5.2). We show first that the elements $[\psi]_\sigma$ for $\psi \in \Phi_f$ are exactly the finite elements in $(\Phi/\sigma; \leq)$. So, assume first that $[\psi]_\sigma$ is finite in $(\Phi/\sigma; \leq)$. By the Support Axiom, $[\psi]_\sigma$ has a support x , hence we may select a representant ψ of the class $[\psi]_\sigma$ with label $d(\psi) = x$. Consider then a directed set X in Φ_x such that its supremum exists in Φ_x and $\psi \leq \bigvee X$. Using Lemma 7.7, we conclude that $[\psi]_\sigma \leq [\bigvee X]_\sigma = \bigvee [X]_\sigma$. Further, the set $[X]_\sigma$ is directed in $(\Phi/\sigma; \leq)$. Since $[\psi]_\sigma$ is finite in $(\Phi/\sigma; \leq)$ there is an element $[\phi]_\sigma$ in $[X]_\sigma$ such that $[\psi]_\sigma \leq [\phi]_\sigma$. But then we may select $\phi \in X$ and $\psi \leq \phi$. This shows that ψ is finite in $(\Phi_x; \leq)$.

Conversely, assume that ψ is finite in $(\Phi_y; \leq)$. Consider a directed set X in $(\Phi/\sigma; \leq)$ such that $[\psi]_\sigma \leq \bigvee X$. Since in compact labeled information algebras $(D; \leq)$ has a greatest element \top , the supremum $\bigvee X$ has support \top . Let $\eta = [\eta]_\sigma = \bigvee X$. Note that any class $[\phi]_\sigma$ has a representant in \top . Define $X' = \{\phi \in \Phi_\top : [\phi]_\sigma \in X\}$. The set X' is directed in $(\Phi_\top; \leq)$ and $\bigvee X'$ exists in Φ_\top . Take further a representant η of the class $[\eta]_\sigma$ in Φ_\top . Then we have $\phi \leq \eta$ for all $\phi \in X'$. We claim that $\eta = \bigvee X'$. In fact, let η' be an upper bound of X' . Then for all $\phi \in X'$, $\phi \leq \eta'$, hence $[\phi]_\sigma \leq [\eta']_\sigma$ and therefore $[\eta]_\sigma \leq [\eta']_\sigma$ so that $\eta \leq \eta'$, hence η is indeed the supremum of X' . Further, we have $t_\top(\psi) \leq \eta$. This implies that there is a $\phi \in X'$ such that $\psi \leq t_\top(\psi) \leq \phi$. It follows that $[\psi]_\sigma \leq [\phi]_\sigma \in X$, which shows that $[\psi]_\sigma$ is finite in $(\Phi/\sigma; \leq)$.

It remains to show strong density. For this purpose consider an element $[\phi]_\sigma = \epsilon_x([\phi]_\sigma)$ in Φ/σ . We take a representant of $[\phi]_\sigma$ with label $d(\phi) = x$. By the local density in the labeled algebra $(\Phi, D; \leq, \perp, \cdot, t)$ we have

$$[\phi]_\sigma = [\bigvee \{\psi \in \Phi_{x,f} : \psi \leq \phi\}]_\sigma.$$

From Lemma 7.7 and the first part of this theorems it follows then that

$$[\phi]_\sigma = \bigvee \{[\psi]_\sigma : [\psi]_\sigma \text{ finite in } (\Phi/\sigma; \leq), [\psi]_\sigma = \epsilon_x([\psi]_\sigma) \leq [\phi]_\sigma\}.$$

This is strong density in the domain-free information algebra $(\Phi/\sigma, D; \leq, \perp, \cdot, \epsilon)$ and this concludes the proof that this algebra is compact. \square

In summary, a domain-free compact information algebra \mathbf{D} transforms into an associated dual labeled compact information algebra \mathbf{DL} . Conversely, a labeled compact information algebra \mathbf{L} has an associated dual domain-free compact information algebra \mathbf{LD} . Then the labeled compact algebra \mathbf{DL} transforms back into the domain-free compact algebra \mathbf{DLD} . Similarly, the domain-free compact algebra \mathbf{LD} transforms back into the labeled compact algebra \mathbf{LDL} . All this holds under the assumption that $(D; \leq)$ has a greatest element \top , what we assume by definition. We have seen in Section 5.3 that \mathbf{D} and \mathbf{DLD} are isomorphic under the map $\psi \mapsto [(\psi, x)]_\sigma$. Similarly, the labeled algebra \mathbf{L} is isomorphic to the algebra \mathbf{LDL} under the

map $\phi \mapsto ([\phi]_\sigma, x)$. We show now that in the case of compact algebras these maps are continuous.

Theorem 7.12 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ and $(\Phi, D; \leq, \perp, \cdot, t)$ be compact domain-free and compact labeled generalised information algebras respectively. Then, if X is a directed subset of Ψ whose supremum exists in Ψ and which has support x ,*

$$[(\bigvee X, x)]_\sigma = \bigvee_{\phi \in X} [(\phi, x)]_\sigma. \quad (7.21)$$

Further, if X is a directed subset of Φ whose supremum exists in Φ and has label x , then

$$([\bigvee X]_\sigma, x) = \bigvee_{\psi \in X} ([\psi]_\sigma, x). \quad (7.22)$$

Proof. We start with (7.21). By Theorem 7.2 we have $\bigvee X = \epsilon_x(\bigvee X) = \bigvee \epsilon_x(X)$. So, using Lemma 7.4

$$[(\bigvee X, x)]_\sigma = [(\bigvee \epsilon_x(X), x)]_\sigma = [\bigvee_{\phi \in X} (\epsilon_x(\phi), x)]_\sigma.$$

From this it follows, using Lemma 7.7,

$$[(\bigvee X, x)]_\sigma = \bigvee_{\phi \in X} [(\epsilon_x(\phi), x)]_\sigma = \bigvee_{\phi \in X} \epsilon_x([\phi]_\sigma).$$

But all elements $[(\phi, x)]_\sigma$ have support x , therefore we conclude

$$[(\bigvee X, x)]_\sigma = \bigvee_{\phi \in X} [(\phi, x)]_\sigma.$$

This is (7.21).

In order to prove (7.22) we note that for $\psi \in X$, we have $\psi \leq \bigvee X$ and $d(\psi) \leq x$. This implies $t_x(\psi) \equiv_\sigma \psi$, hence $\epsilon_x([\psi]_\sigma) = [t_x(\psi)]_\sigma = [\psi]_\sigma$. So, x is a support for all $[\psi]_\sigma$ such that $\psi \in X$. Define $X' = \{t_x(\psi) : \psi \in X\}$. Then, by Lemma 7.6, $\bigvee X = \bigvee X' = \bigvee_{\psi \in X} t_x(\psi)$. Therefore, we see that (Lemma 7.7)

$$[\bigvee X]_\sigma = [\bigvee X']_\sigma = [\bigvee_{\psi \in X} t_x(\psi)]_\sigma = \bigvee_{\psi \in X} [t_x(\psi)]_\sigma = \bigvee_{\psi \in X} [\psi]_\sigma \quad (7.23)$$

So, from Lemma 7.4 we obtain

$$([\bigvee X]_\sigma, x) = (\bigvee_{\psi \in X} [\psi]_\sigma, x) = \bigvee_{\psi \in X} ([\psi]_\sigma, x).$$

This is (7.22). \square

As remarked above, this theorem shows that $\mathbf{D} \cong \mathbf{DLD}$ and $\mathbf{L} \cong \mathbf{LDL}$ under *continuous* isomorphisms, if \mathbf{D} and \mathbf{L} are compact domain-free or labeled information algebras respectively.

We now turn to the algebraic case. What is an algebraic labeled information algebra, and what can be said about the duality between algebraic domain-free and labeled algebras? We start with an algebraic domain-free generalised information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ and examine its dual labeled version $(\Phi, D; \leq, \perp, \cdot, t)$, where Φ is, as always, the set of all pairs (ψ, x) with $\psi \in \Psi$ and $\epsilon_x(\psi) = \psi$. Consider a subset X of Φ . If its supremum $\bigvee X$ exists in Φ , then $\bigvee X = (\bigvee X', \bigvee X'')$, where $X' = \{\psi : (\psi, x) \in X\}$ and $X'' = \{x : (\psi, x) \in X\}$ (Lemma 7.3). Since $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is algebraic, $\bigvee X'$ exists always. However, there is no guarantee that $\bigvee X''$ exists. So, even if $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is algebraic, this does not imply, that any subset X of Φ has a supremum, $(\Phi; \leq)$ is not necessarily complete. However, for any $x \in D$, the local orders $(\Phi_x; \leq)$ are complete. If X is a subset of Φ_x , then $\bigvee X = (\bigvee X', x)$. So, $(\Phi, D; \leq, \perp, \cdot, t)$ is a compact labeled information algebra, which is locally complete in this sense. Alternatively, $(\Phi_x; \leq)$ is a dcpo, which together with compactness implies that it is a complete lattice. This leads to the following definition:

Definition 7.5 *A labeled generalised information algebra $(\Phi, D; \leq, \perp, \cdot, t)$ is called algebraic if*

1. *it is compact,*
2. *for all $x \in D$, $(\Phi_x; \leq)$ is a dcpo.*

Recall that in a compact labeled information algebra, $(D; \leq)$ is assumed to have a greatest element \top . Note further that in an algebraic domain-free information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ we may always, as in a compact information algebra, adjoin a top domain, if $(D; \leq)$ has not already a greatest element. Therefore, in its labeled version we may likewise without loss of generality assume that D has a greatest element \top . Then Φ_\top contains the pairs (ψ, \top) for all $\psi \in \Psi$ and $(\Phi_\top; \leq)$ is essentially the same as $(\Psi; \leq)$ as we shall see below (Theorem 7.14). Then, the domain-free version of a labeled algebraic information algebra is also algebraic, as the following theorem shows.

Theorem 7.13 *Let $(\Phi, D; \leq, \perp, \cdot, t)$ be an algebraic labeled information algebra. Then $(\Phi/\sigma, D; \leq, \perp, \cdot, \epsilon)$ is an algebraic domain-free information algebra.*

Proof. From Theorem 7.11 we know that $(\Phi/\sigma, D; \leq, \perp, \cdot, \epsilon)$ is a compact domain-free information algebra. It remains to show that $(\Phi/\sigma; \leq)$ is a dcpo.

Consider a directed subset X of Φ/σ . Note that for all $\phi \in \Phi$, $\phi \equiv_\sigma t_\top(\phi)$. Therefore, we may always take the representant of the class $[\phi]_\sigma$ in the top domain. Define

$$X' = \{\phi \in \Phi_\top : [\phi]_\sigma \in X\}.$$

The supremum $\bigvee X'$ exists in Φ_\top since $(\Phi, D; \leq, \perp, \cdot, t)$ is algebraic. By Lemma 7.7 we obtain then $[\bigvee X']_\sigma = \bigvee_{\phi \in X'} [\phi]_\sigma = \bigvee X$. So the supremum of X exists in Φ/σ and $(\Phi/\sigma; \leq)$ is a dcpo. \square

The proof shows that somehow the whole domain-free information algebra is already incorporated in the top-level domain of the labeled algebra. This can be made more precise: Define in Φ_\top the following operations:

1. *Combination*: $\phi, \psi \in \Phi_\top \mapsto \phi \cdot \psi$, where \cdot denotes the combination in Φ ,
2. *Extraction*: $\phi \in \Phi_\top, x \in D \mapsto \epsilon_x(\phi) = t_\top(t_x(\phi))$.

We claim that with these operations, $(\Phi_\top, D; \leq, \perp, \cdot, \epsilon)$ becomes a domain-free information algebra.

Theorem 7.14 *Let $(\Phi, D; \leq, \perp, \cdot, t)$ be a labeled information algebra and assume that D has a greatest element \top . Then $(\Phi_\top, D; \leq, \perp, \cdot, \epsilon)$ with the operations of combination and extraction as defined above is a domain-free information algebra, isomorphic to $(\Phi/\sigma, D; \leq, \perp, \cdot, \epsilon)$. If $(\Phi, D; \leq, \perp, \cdot, t)$ is algebraic, then $(\Phi_\top, D; \leq, \perp, \cdot, \epsilon)$ is so too.*

Proof. The q-separoid and semigroup axioms A0 and A1 are evident. Clearly, \top is a support for all $\phi \in \Phi_\top$. If x is a support for $\phi \in \Phi_\top$, that is, $\phi = t_\top(t_x(\phi))$, and $y \geq x$, then, using elementary properties of transport (Lemma 3.1), we obtain

$$\begin{aligned} \epsilon_y(\phi) &= t_\top(t_y(\phi)) = t_\top(t_y(t_\top(t_x(\phi)))) = t_\top(t_y(t_\top(t_y(t_x(\phi))))) \\ &= t_\top(t_y(t_x(\phi))) = t_\top(t_x(\phi)) = \phi. \end{aligned}$$

This shows that the Support Axiom A2 is valid. Further, 1_\top is the unity of combination in Φ_\top and 0_\top is the null element. Obviously $\epsilon_x(1_\top) = t_\top(t_x(1_\top)) = 1_\top$ and similarly $\epsilon_x(0_\top) = t_\top(t_x(0_\top)) = 0_\top$. Further, if $\epsilon_x(\phi) = t_\top(t_x(\phi)) = 0_\top$, then $\phi = 0_\top$. This is axiom A3.

Assume now that $x \perp y | z$ and $\epsilon_x(\phi) = \phi$. Then

$$\epsilon_y(\epsilon_z(\phi)) = t_\top(t_y(t_\top(t_z(\phi)))) = t_\top(t_y(t_z(\phi))),$$

since a transport from z to y can always pass by the larger domain \top . Further,

$$t_\top(t_y(t_z(\phi))) = t_\top(t_y(t_z(t_\top(t_x(\phi))))) = t_\top(t_y(t_z(t_x(\phi)))) = t_\top(t_y(t_x(\phi)))$$

by the Transport Axiom in the labeled algebra. We conclude that

$$t_{\top}(t_y(t_z(\phi))) = t_{\top}(t_y(t_{\top}(t_x(\phi)))) = t_{\top}(t_y(\phi)).$$

This shows that $\epsilon_y(\epsilon_z(\phi)) = \epsilon_y(\phi)$, thus the Extraction Axiom A4 holds.

If $x \perp y | z$ and $\epsilon_x(\phi) = \phi$, $\epsilon_y(\psi) = \psi$, then

$$\begin{aligned} \epsilon_z(\phi \cdot \psi) &= t_{\top}(t_z(\phi \cdot \psi)) = t_{\top}(t_z(t_{\top}(t_x(\phi)) \cdot t_{\top}(t_y(\psi)))) \\ &= t_{\top}(t_z(t_{\top}(t_x(\phi)) \cdot t_y(\psi))) \\ &= t_{\top}(t_z(t_x(\phi) \cdot t_y(\psi))) = t_{\top}(t_z(t_x(\phi)) \cdot t_z(t_y(\psi))) \end{aligned}$$

since the transport of a combination to a larger domain equals the combination of the transports to the larger domain, and by the Combination Axiom of the labeled algebra. This implies further that

$$\begin{aligned} \epsilon_z(\phi \cdot \psi) &= t_{\top}(t_z(t_{\top}(t_x(\phi))) \cdot t_z(t_{\top}(t_y(\psi)))) \\ &= t_{\top}(t_z(\phi) \cdot t_z(\psi)) = t_{\top}(t_z(\phi)) \cdot t_{\top}(t_z(\psi)) \\ &= \epsilon_z(\phi) \cdot \epsilon_z(\psi). \end{aligned}$$

This verifies the validity of the Combination Axiom A5.

It remains to verify Idempotency. Since $\phi = t_{\top}(\phi)$, we have indeed $\phi \cdot \epsilon_x(\phi) = \phi \cdot t_{\top}(t_x(\phi)) = t_{\top}(\phi) \cdot t_{\top}(t_x(\phi)) = t_{\top}(\phi \cdot t_x(\phi)) = t_{\top}(\phi) = \phi$ by idempotency in the labeled algebra. So $(\Phi_{\top}, D; \leq, \perp, \cdot, \epsilon)$ is a domain-free idempotent generalised information algebra.

The map $\phi \in \Phi_{\top} \mapsto [\phi]_{\sigma}$ is clearly one-to-one and onto Φ/σ . It is also a homomorphism since $[\phi \cdot \psi]_{\sigma} = [\phi]_{\sigma} \cdot [\psi]_{\sigma}$ and $[t_{\top}(t_x(\phi))]_{\sigma} = [t_x(\phi)]_{\sigma} = \epsilon_x([\phi]_{\sigma})$.

If $(\Phi, D; \leq, \perp, \cdot, t)$ is algebraic, then $(\Phi_{\top}; \leq)$ is a complete lattice. But this means that $(\Phi_{\top}, D; \leq, \perp, \cdot, \epsilon)$ is algebraic. The isomorphism is in this case continuous (see Lemma 7.7). \square

In developing the duality of compact and algebraic information algebras above, the existence of a greatest element in $(D; \leq)$ was crucial. It is not clear, whether duality can also be obtained without this somewhat artificial assumption.

7.4 Continuous Algebras

The notion of approximation can be somewhat weakened. This leads to a generalisation of the concept of compact information algebras. The present section is partially based on (Guan & Li, 2012) but applies to generalised information algebras. The basic notion in this section is the way-below relation in an ordered set.

Definition 7.6 Way-Below. Let $(\Psi; \leq)$ be a partially ordered set. For $\phi, \psi \in \Psi$ we write $\psi \ll \phi$ and say ψ is way-below ϕ , if for every directed

set $X \subseteq \Psi$, for which the supremum exists, $\phi \leq \bigvee X$ implies that there is an element $\chi \in X$ such that $\psi \leq \chi$.

Note that ϕ is a finite element if and only if $\phi \ll \phi$. The following lemma lists some well-known elementary results on the way-below relation, see for instance (Gierz, 2003).

Lemma 7.8 *Let $(\Psi; \leq)$ be a partially ordered set. Then the following holds for $\phi, \psi \in \Psi$*

1. $\psi \ll \phi$ implies $\psi \leq \phi$,
2. $\psi \ll \phi$ and $\phi \leq \chi$ imply $\psi \ll \chi$,
3. $\chi \leq \psi$ and $\psi \ll \phi$ imply $\chi \ll \phi$.
4. $\chi \ll \psi$ and $\psi \ll \phi$ imply $\chi \ll \phi$.

We are of course interested in the way-below relation in case that $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is a domain-free idempotent information algebra, that is, Ψ is a semi-lattice. Then the way-below relation has some additional properties.

Lemma 7.9 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be a domain-free generalised information algebra. Then*

1. $1 \ll \phi$ for all $\phi \in \Psi$.
2. $\psi_1, \psi_2 \ll \phi$ implies $\psi_1 \vee \psi_2 = \psi_1 \cdot \psi_2 \ll \phi$ for all $\psi_1, \psi_2 \in \Psi$.
3. The set $\{\psi \in \Psi : \psi \ll \phi\}$ is an ideal for all $\phi \in \Psi$.
4. $\psi \ll \phi$ if and only if for all $X \subseteq \Psi$ such that $\bigvee X$ exists and $\phi \leq \bigvee X$, there is a finite subset F of X such that $\psi \leq \bigvee F$.

Proof. (1) Let $X \subseteq \Psi$ be a directed set, and $\phi \leq \bigvee X$. Since X is non-empty, there is a $\psi \in X$ and $1 \leq \psi$, hence $1 \ll \phi$.

(2) Assume $\psi_1, \psi_2 \ll \phi$. Consider any directed set $X \subseteq \Psi$ such that $\phi \leq \bigvee X$. Then there exist elements $\chi_1, \chi_2 \in X$ so that $\psi_1 \leq \chi_1$ and $\psi_2 \leq \chi_2$. Since X is directed, there is also an element $\chi \in X$ so that $\chi_1, \chi_2 \leq \chi$. But then, $\psi_1 \vee \psi_2 \leq \chi_1 \vee \chi_2 \leq \chi$. This shows that $\psi_1 \vee \psi_2 \ll \phi$.

(3) Assume $\psi \ll \phi$ and $\chi \leq \psi$. Then by Lemma 7.8 (3) $\chi \ll \phi$. Further let $\psi_1 \ll \phi$ and $\psi_2 \ll \phi$. By (2) just proved, $\psi_1 \vee \psi_2 \ll \phi$. Hence $\{\psi \in \Psi : \psi \ll \phi\}$ is an ideal.

(4) Suppose first that $\psi \ll \phi$. Let X be a subset of Ψ such that $\bigvee X$ exists and $\phi \leq \bigvee X$. Let Y be the set of all joins of finite subsets of X . Then $X \subseteq Y$ and $\bigvee X$ is an upper bound for Y . Let χ be another upper bound of Y . Then χ is an upper bound of X , hence $\bigvee X \leq \chi$. So $\bigvee X$ is

the supremum of Y , $\bigvee X = \bigvee Y$. Furthermore Y is a directed set. So there is an element $\eta \in Y$ such that $\psi \leq \eta$ and $\eta = \bigvee F$ for some finite subset F of X .

Conversely, consider elements $\psi, \phi \in \Psi$ such that condition 4 of the lemma holds. Let X be a directed subset of Ψ such that $\bigvee X$ exists and $\phi \leq \bigvee X$. There is a finite subset F of X such that $\psi \leq \bigvee F$. Since X is directed, there is a $\chi \in X$ such that $\bigvee F \leq \chi$, hence $\psi \leq \chi$. So $\psi \ll \phi$. \square

With the aid of the way-below relation, algebraic information algebras can be alternatively characterized.

Theorem 7.15 *If $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is an idempotent domain-free information algebra, then the following conditions are equivalent:*

1. $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is a domain-free algebraic information algebra with finite elements Ψ_f .
2. $(\Psi; \leq)$ is an algebraic lattice with finite elements Ψ_f and $\forall x \in D, \forall \phi \in \Psi$

$$\epsilon_x(\phi) = \bigvee \{\psi \in \Psi_f : \psi = \epsilon_x(\psi) \ll \phi\}. \quad (7.24)$$

Proof. (1) \Rightarrow (2): By definition $(\Psi; \leq)$ is an algebraic lattice, that is a complete lattice with finite elements Ψ_f . Then condition (7.24) follows from strong density and Lemma 7.8 in the following way,

$$\begin{aligned} \epsilon_x(\phi) &= \bigvee \{\psi \in \Psi_f : \psi = \epsilon_x(\psi) \leq \phi\} \\ &= \bigvee \{\psi : \psi \ll \psi = \epsilon_x(\psi) \leq \phi\} \\ &= \bigvee \{\psi : \psi \ll \psi = \epsilon_x(\psi) \ll \phi\} \\ &= \bigvee \{\psi \in \Psi_f : \psi = \epsilon_x(\psi) \ll \phi\}. \end{aligned}$$

(2) \Rightarrow (1): We use Theorem 7.3 and take Ψ_f for Ψ' . Convergence holds, since $(\Psi; \leq)$ is a complete lattice, density follows from (7.24) since $\psi \ll \phi$ implies $\psi \leq \phi$ and compactness follows from the lattice-theoretic finiteness. \square

Another important property of finite elements in a compact information algebra is given by the following theorem:

Theorem 7.16 *If $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is a compact domain-free information algebra, then $\psi \ll \phi$ implies that there is an element $\chi \in \Psi_f$ so that $\psi \leq \chi \leq \phi$.*

Proof. The set $A_\phi = \{\chi \in \Psi_f : \chi \leq \phi\}$ is directed and $\phi = \bigvee A_\phi$, hence $\phi \leq \bigvee A_\phi$. Then $\psi \ll \phi$ implies the existence of an element $\chi \in A_\phi$ so that $\psi \leq \chi$. But $\chi \leq \phi$. So $\psi \leq \chi \leq \phi$ and $\chi \in \Psi_f$. \square

A set of elements having the property that $\psi \ll \phi$ implies the existence of a $\chi \in S$ such that $\psi \leq \chi \leq \phi$ is called *separating*. So the set of finite elements in a compact information algebra is separating.

We now introduce continuous information algebras and show that they are a generalisation of algebraic ones.

Definition 7.7 Continuous Information Algebras. A generalised domain-free information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is called *continuous with basis* $B \subseteq \Psi$ if B is closed under join (combination), contains the unit 1, and B satisfies the following conditions:

1. *Convergence:* If $X \subseteq B$ is directed, then $\bigvee X$ exists in Ψ .
2. *B-Density:* For all $\phi \in \Psi$ and for all $x \in D$,

$$\epsilon_x(\phi) = \bigvee \{\psi \in B : \psi = \epsilon_x(\psi) \ll \epsilon_x(\phi)\}.$$

Note that in a compact information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ the finite elements Ψ_f form a basis. So, an algebraic information algebra is also continuous with basis Ψ_f . We shall present below an example of a continuous information algebra which is not compact. So continuous information algebras present a genuine generalisation of compact information algebras. The approximation by finite elements is replaced by an approximation of some more general elements in a basis B .

Strong B -density implies weak B -density: In fact let $\phi \in \Psi$, then there is a $x \in D$ so that $\phi = \epsilon_x(\phi)$. Then by the strong B -density:

$$\begin{aligned} \phi &= \epsilon_x(\phi) = \bigvee \{\psi \in B : \psi = \epsilon_x(\psi) \ll \phi\} \\ &\leq \bigvee \{\psi \in B : \psi \ll \phi\} \leq \phi. \end{aligned}$$

This is weak B -density. This allows again to adjoin the top domain \top to $(D; \leq)$ in a continuous information algebra, since weak B -density is also local B -density on the domain \top .

For later purposes we remark that both the sets $\{\psi \in B : \psi \ll \phi\}$ and $\{\psi \in B : \psi = \epsilon_x(\psi) \ll \phi\}$ are directed. Note also that $\psi \ll \phi$ does not imply $\epsilon_x(\psi) \ll \epsilon_x(\phi)$.

Just as in an algebraic information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$, the partial order $(\Psi; \leq)$ is an algebraic lattice, it follows that in a continuous information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ the partial order $(\Psi; \leq)$ is a *continuous lattice*, namely a complete lattice such that for all $\phi \in \Psi$

$$\phi = \bigvee \{\psi \in \Psi : \psi \ll \phi\}. \quad (7.25)$$

The following theorem states the situation more precisely.

Theorem 7.17 *If $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is an idempotent domain-free information algebra, then the following are equivalent:*

1. $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is a continuous information algebra.
2. $(\Psi; \leq)$ is a continuous lattice, and $\forall x \in D, \forall \phi \in \Psi$.

$$\epsilon_x(\phi) = \bigvee \{\psi \in \Psi : \psi = \epsilon_x(\psi) \ll \epsilon_x(\phi)\}. \quad (7.26)$$

Proof. Assume first $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ to be a continuous information algebra with basis B . We show first that $(\Psi; \leq)$ is a complete lattice. Consider a non-empty subset X of Ψ . Define Y to be the set of all elements in B , which are way-below all elements in X ,

$$Y = \{\psi \in B : \psi \ll \phi \text{ for all } \phi \in X\}.$$

Since $1 \in Y$, the set is non-empty, and with $\psi_1, \psi_2 \in Y$ also $\psi_1 \vee \psi_2 \in Y$ (Lemma 7.9). So the set Y is directed. Therefore $\bigvee Y$ exists and is a lower bound of X . Assume ψ to be another lower bound of X . Then $A_\psi = \{\eta \in B : \eta \ll \psi\} \subseteq Y$, since $\eta \ll \psi \leq \phi$ implies $\eta \ll \phi$. From this we conclude that $\psi = \bigvee A_\psi \leq \bigvee Y$, hence $\bigvee Y$ is the infimum of X . Further, since Ψ has a top element 0 it follows from standard results of lattice theory, that $(\Psi; \leq)$ is a complete lattice. Further, using B -density, we obtain for all $\phi \in \Psi$,

$$\phi = \bigvee \{\psi \in B : \psi \ll \phi\} \leq \bigvee \{\psi \in \Psi : \psi \ll \phi\} \leq \phi.$$

So $(\Psi; \leq)$ is indeed a continuous lattice. Further, again by density,

$$\begin{aligned} \epsilon_x(\phi) &= \bigvee \{\psi \in B : \psi = \epsilon_x(\psi) \ll \epsilon_x(\phi)\} \\ &\leq \bigvee \{\psi \in \Psi : \psi = \epsilon_x(\psi) \ll \epsilon_x(\phi)\} \leq \epsilon_x(\phi), \end{aligned}$$

so (7.26) holds.

If $(\Psi; \leq)$, on the other hand, is a complete lattice, then convergence holds with Ψ as a basis. And (7.26) is exactly B -density with respect to the basis Ψ . Hence $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is a continuous information algebra. \square

Here follows an example of a continuous information algebra.

Example 7.2 Continuous Valuation Algebra: This example is from (Guan & Li, 2012). Let $\Psi = [0, 1]$ be the real interval between 0 and 1 and $D = \{0, 1\}$. Join is defined as maximum, the number 0 is the unit and the number 1 the null element of the algebra. Information extraction is defined as follows:

$$\begin{aligned} \epsilon_1(\phi) &= \phi, \\ \epsilon_0(\phi) &= \begin{cases} \phi & \text{if } \phi \in [0, 1/2], \\ 1/2 & \text{if } \phi \in (1/2, 1]. \end{cases} \end{aligned}$$

We leave it to reader to verify the axioms of an idempotent valuation algebra.

Any non-empty subset X of $[0, 1]$ is in this example directed and $\sup X$ exists always. The relation $\psi \ll \phi$ holds if either $0 < \psi < \phi$ or in particular if $\psi = \phi = 0$. As a basis we take $B = \Psi$. Then it can be verified that $\epsilon_x(\phi) = \bigvee \{\psi \in B : \psi = \epsilon_x(\psi) \ll \phi\}$ holds both for $x = 0$ and $x = 1$. So it is a continuous information algebra. But it is not compact: The only element satisfying $\phi \ll \phi$ is $\phi = 0$. \ominus

We have seen above that a compact information algebra is continuous. But the converse does not hold as the example above shows. Here follows a necessary and sufficient condition for a continuous information algebra to be algebraic.

Theorem 7.18 *A continuous domain-free information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is algebraic, if and only if the set $\{\phi \in \Psi : \phi \ll \phi\}$ is a basis for $(\Psi, D; \leq, \perp, \cdot, \epsilon)$.*

Proof. We know already that if $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is algebraic, then it is continuous, with basis $B = \Psi_f = \{\phi \in \Psi : \phi \ll \phi\}$.

So, assume that $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is continuous with basis $B = \{\phi \in \Psi : \phi \ll \phi\}$. The lattice $(\Psi; \leq)$ is complete, hence it is a dcpo. Strong density is derived as follows:

$$\begin{aligned} \epsilon_x(\phi) &= \bigvee \{\psi \in B : \psi = \epsilon_x(\psi) \ll \epsilon_x(\phi)\} \\ &= \bigvee \{\psi \in B : \psi = \epsilon_x(\psi) \ll \psi \leq \epsilon_x(\phi)\} \\ &= \bigvee \{\psi \in B : \psi = \epsilon_x(\psi) \leq \epsilon_x(\phi)\} \end{aligned} \tag{7.27}$$

So, the algebra is compact, hence algebraic with the set $\{\phi \in \Psi : \phi \ll \phi\}$ as finite elements. \square

The following Theorem gives another necessary and sufficient condition for an information algebra to be continuous.

Theorem 7.19 *An idempotent domain-free generalised information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is continuous if and only if,*

1. $(\Psi; \leq)$ is a continuous lattice,
2. for all $x \in D$ and any directed set $X \subset \Psi$,

$$\epsilon_x(\bigvee X) = \bigvee_{\phi \in X} \epsilon_x(\phi). \tag{7.28}$$

Proof. Assume $(\Psi; \leq)$ to be a continuous lattice, so that weak density holds (7.25), and that (7.28) holds too. Then $(\Psi; \leq)$ is a complete lattice. Consider a $\phi \in \Psi$ and let $\phi' = \epsilon_x(\phi)$. Then by weak density $\phi' = \bigvee \{\psi \in \Phi : \psi \ll \phi'\}$, and $\{\psi \in \Phi : \psi \ll \phi'\}$ is a directed set. From this we deduce, using (7.28)

$$\begin{aligned} \epsilon_x(\phi) &= \epsilon_x(\epsilon_x(\phi)) = \epsilon_x(\bigvee \{\psi \in \Psi : \psi \ll \epsilon_x(\phi)\}) \\ &= \bigvee \{\epsilon_x(\psi) : \psi \ll \epsilon_x(\phi)\}. \end{aligned}$$

Let $\eta = \epsilon_x(\psi)$ so that $\eta = \epsilon_x(\eta) \leq \psi \ll \epsilon_x(\phi)$. From this it follows that $\eta \ll \epsilon_x(\phi)$ and therefore,

$$\begin{aligned} \epsilon_x(\phi) &= \bigvee \{\eta : \eta = \epsilon_x(\eta) = \epsilon_x(\psi), \psi \ll \epsilon_x(\phi)\} \\ &\leq \bigvee \{\eta : \eta = \epsilon_x(\eta) \ll \epsilon_x(\phi)\} \leq \epsilon_x(\phi). \end{aligned}$$

Hence we have $\epsilon_x(\phi) = \bigvee \{\eta : \eta = \epsilon_x(\eta) \ll \epsilon_x(\phi)\}$ and by Theorem 7.17 $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is a continuous information algebra.

Conversely, assume $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ to be a continuous information algebra with basis B . Then $(\Psi; \leq)$ is a continuous, hence complete lattice (Theorem 7.17). Consider a directed set $X \subseteq \Psi$ and $x \in D$. For $\phi \in X$ we have $\phi \leq \bigvee X$, hence $\epsilon_x(\phi) \leq \epsilon_x(\bigvee X)$ and therefore $\bigvee_{\phi \in X} \epsilon_x(\phi) \leq \epsilon_x(\bigvee X)$. By strong B -density,

$$\epsilon_x(\bigvee X) = \bigvee \{\psi \in B : \psi = \epsilon_x(\psi) \ll \epsilon_x(\bigvee X)\}.$$

Now, $\psi = \epsilon_x(\psi) \ll \epsilon_x(\bigvee X) \leq \bigvee X$ implies that there is a $\phi \in X$ so that $\psi \leq \phi$ and thus also $\psi = \epsilon_x(\psi) \leq \epsilon_x(\phi)$. From this we conclude that $\epsilon_x(\bigvee X) \leq \bigvee_{\phi \in X} \epsilon_x(\phi)$ and thus $\epsilon_x(\bigvee X) = \bigvee_{\phi \in X} \epsilon_x(\phi)$. Hence (7.28) is valid. \square

What is the labeled version of a continuous labeled information algebra? To examine this question, we consider the labeled version $(\Phi, D; \leq, \perp, \cdot, t)$ of a continuous information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. We remind that Φ consists of all pairs (ψ, x) , where $\psi \in \Psi$ and $\psi = \epsilon_x(\psi)$.

Assume that B is a basis of the continuous information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. Define $B_x = \{(\psi, x) : \psi \in B, \epsilon_x(\psi) = \psi\}$. We claim that this is a basis in Ψ_x . In fact, if $(\phi, x), (\psi, x) \in B_x$, then $(\phi, x) \cdot (\psi, x) = (\phi \cdot \psi, x) \in B_x$ since B is closed under combination or join. So B_x is closed under combination. Further also $(1, x)$ belongs to B . Consider any directed subset X of B_x . By Lemma 7.4 we have $\bigvee X = (\bigvee_{(\psi, x) \in X} \psi, x) \in \Phi_x$. This is the convergence property in Φ_x .

Define

$$\bar{B} = \bigcup_{x \in D} B_x.$$

Then, \bar{B} is still closed under combination. In fact, let $(\phi, x) \in B_x$ and $(\psi, y) \in B_y$, then $\phi, \psi \in B$ and x is a support of ϕ , y a support of ψ . But then $x \vee y$ is a support of $\phi \cdot \psi$. So, since $(\phi, x) \cdot (\psi, y) = (\phi \cdot \psi, x \vee y)$ and $\phi \cdot \psi \in B$, we see that $(\phi, x) \cdot (\psi, y) \in B_{x \vee y}$.

We claim also that a *density* property holds in Ψ_x . Denote the way-below relation in $(\Phi_x; \leq)$ by \ll_x . We prove first the following lemma.

Lemma 7.10 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be a continuous domain-free information algebra and let $\phi, \psi \in \Psi$ and $\epsilon_x(\phi) = \phi$, $\epsilon_x(\psi) = \psi$. Then $\psi \ll \phi$, if and only if $(\psi, x) \ll_x (\phi, x)$.*

Proof. Assume $\psi \ll \phi$ and $\epsilon_x(\phi) = \phi$, $\epsilon_x(\psi) = \psi$. Consider a directed set $X \subseteq \Phi_x$. Then $X' = \{(\phi, x) : (\phi, x) \in X\}$ is directed too. Now, $(\phi, x) \leq \bigvee X$ implies $\phi \leq \bigvee X'$. Then, there is a $\chi \in X'$ such that $\psi \leq \chi$. Note that $\epsilon_x(\chi) = \chi$. Hence we see that $(\psi, x) \leq (\chi, x)$. So indeed $(\phi, x) \ll_x (\psi, x)$.

Conversely, assume $(\psi, x) \ll_x (\phi, x)$. Consider a directed set $X \subseteq \Psi$ such that $\phi \leq \bigvee X$. In a continuous information algebra we have $\epsilon_x(\bigvee X) = \bigvee_{\phi \in X} \epsilon_x(\phi)$ (Theorem 7.19). Then $\phi = \epsilon_x(\phi) \leq \epsilon_x(\bigvee X) = \bigvee_{\chi \in X} \epsilon_x(\chi)$. Therefore $(\phi, x) \leq (\bigvee_{\chi \in X} \epsilon_x(\chi), x) = \bigvee_{\chi \in X} (\epsilon_x(\chi), x)$ (Lemma 7.4). Since the set $\{(\epsilon_x(\chi), x) : \chi \in X\}$ is directed, there must then be a $\chi \in X$ such that $(\psi, x) \leq (\epsilon_x(\chi), x)$. Then $\psi = \epsilon_x(\psi) \leq \epsilon_x(\chi) \leq \chi \in X$. This proves that $\psi \ll \phi$. \square

This allows us to derive density, using Lemma 7.4 and Lemma 7.10 in (Φ_x, \leq) ,

$$\begin{aligned} & \bigvee \{(\psi, x) \in B_x : (\psi, x) \ll_x (\phi, x)\} \\ &= (\bigvee \{\psi : \psi \in B, \psi = \epsilon_x(\psi) \ll \phi = \epsilon_x(\phi)\}, x) \\ &= (\phi, x). \end{aligned}$$

This is the density property claimed above.

Finally, assume $(\psi, x) \ll_x (\phi, x)$. By Lemma 7.10 we have $\psi \ll \phi$ and x is a support of both ψ and ϕ . If $x \leq y$, then y is also a support of both elements. Therefore, again by Lemma 7.10, we have that $t_y(\psi, x) = (\psi, y) \ll_y (\phi, y) = t_y(\phi, x)$. Conversely, assume that x is a support of ψ and ϕ and $x \leq y$. Then, if $(\psi, y) \ll_y (\phi, y)$, Lemma 7.10 implies that $\psi \ll \phi$, hence $(\psi, x) \ll_x (\phi, x)$. This is an important *compatibility* relation between the way-below relation in different domains Ψ_x and Ψ_y .

We summarise these results in the following theorem.

Theorem 7.20 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be a continuous domain-free information algebra with basis B and $(\Phi, D; \leq, \perp, \cdot, t)$ the associated dual labeled information algebra. Then the following properties hold:*

1. B_x is a basis in $(\Phi_x; \leq)$, that is B_x is closed under combination and contains $(1, x)$. Any directed subset of B_x has a supremum in Φ_x .

2. $(\phi, x) = \bigvee \{(\psi, x) \in B_x : (\psi, x) \ll_x (\phi, x)\}$, for all $(\phi, x) \in \Psi_x$.
3. If $x \leq y$, then $(\psi, x) \ll_x (\phi, x)$ if and only if $t_y(\psi, x) \ll_y t_y(\phi, x)$.

This theorem serves as a base to define the concept of a labeled continuous information algebra.

Definition 7.8 Labeled Continuous Information Algebra: A labeled idempotent generalised information algebra $(\Phi, D; \leq, \perp, \cdot, t)$ is called continuous, if there is for all $x \in D$ a set $B_x \subseteq \Psi_x$ (the basis in x), closed under combination and contains 1_x , satisfying the following conditions for all $x \in D$:

1. Convergence: If $X \subseteq B_x$ is directed, then $\bigvee X \in \Psi_x$.
2. Density: For all $\phi \in \Phi_x$, $\phi = \bigvee \{\psi \in B_x : \psi \ll_x \phi\}$.
3. Compatibility: If $d(\phi) = d(\psi) = x \leq y$, then $\psi \ll_x \phi$ if and only if $t_y(\psi) \ll_y t_y(\phi)$.

According to this definition and Theorem 7.20, the dual labeled information algebra $(\Phi, D; \leq, \perp, \cdot, t)$ associated with a continuous domain-free information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is itself continuous. We remark that, as in Theorem 7.17, it follows that $(\Phi_x; \leq)$ is a continuous lattice for every $x \in D$.

To establish duality for continuous information algebras, let's start with a labeled continuous information algebra $(\Phi, D; \leq, \perp, \cdot, t)$ and consider its associated dual domain-free information algebra $(\Phi/\sigma, D; \leq, \perp, \cdot, \epsilon)$. Is this algebra continuous too? A conditionally affirmative answer is given by Theorem 7.21 below. In order to prove this theorem we need two auxiliary results, which have some interest by themselves.

Lemma 7.11 *Let $(\Phi, D; \leq, \perp, \cdot, t)$ be an idempotent labeled generalised information algebra. Then $\epsilon_x([\psi]_\sigma) = [\psi]_\sigma \ll [\phi]_\sigma = \epsilon_x([\phi]_\sigma)$ in Φ/σ implies $\psi \ll_x \phi$ for the representants ψ and ϕ of $[\psi]_\sigma$ and $[\phi]_\sigma$ with $d(\psi) = d(\phi) = x$. Further, if D has a top element \top , and $(\Phi, D; \leq, \perp, \cdot, t)$ is labeled continuous, then, if $d(\psi) = d(\phi) = x$, $\psi \ll_x \phi$ implies $[\psi]_\sigma \ll [\phi]_\sigma$.*

Proof. Assume $X \subseteq \Phi_x$ directed, $\phi, \psi \in \Phi_x$ representants of the classes $[\phi]_\sigma$ and $[\psi]_\sigma$ respectively and $\phi \leq \bigvee X$. Then $[\phi]_\sigma \leq \bigvee [X]_\sigma$ with $[X]_\sigma = \{[\chi]_\sigma : \chi \in X\}$ (Lemma 7.7). The set $[X]_\sigma$ is directed, therefore $[\psi]_\sigma \ll [\phi]_\sigma$ implies that there is a $\eta \in X$ such that $[\psi]_\sigma \leq [\eta]_\sigma$, hence $\psi \leq \eta$. This proves that $\psi \ll_x \phi$.

For the second part, assume first $\psi \ll_\top \phi$ and consider a directed set X in Φ/σ such that $[\phi]_\sigma \leq \bigvee X$. We may take as representants of the classes $[\eta]_\sigma$ in the set X their representants in Φ_\top . Let then $X' = \{\eta \in \Phi_\top : [\eta]_\sigma \in X\}$.

X' is still directed. Now, if $[\phi]_\sigma \leq \bigvee X$ and ϕ is again a representant of $[\phi]_\sigma$ in Φ_\top , then also $\phi \leq \bigvee X'$. Since $\psi \ll_\top \phi$, there is an element $\eta \in X'$ such that $\psi \leq \eta$. But then $[\eta]_\sigma \in X$ and $[\psi]_\sigma \leq [\eta]_\sigma$. This shows that $[\psi]_\sigma \ll [\phi]_\sigma$. Now, if $d(\psi) = d(\phi) = x$ and $\psi \ll_x \phi$, then by the compatibility property $t_\top(\psi) \ll_\top t_\top(\phi)$, and $[\psi]_\sigma = [t_\top(\psi)]_\sigma$ and $[\phi]_\sigma = [t_\top(\phi)]_\sigma$, hence $[\psi]_\sigma \ll [\phi]_\sigma$ as just proved. \square

The next lemma is similar as Lemma 7.6 for labeled compact algebras.

Lemma 7.12 *Let $(\Phi, D; \leq, \perp, \cdot, t)$ be a labeled continuous information algebra. If $X \subseteq \Psi_y$ directed, then for all $x \leq y \in D$,*

$$t_x(\bigvee X) = \bigvee t_x(X), \quad (7.29)$$

where $t_x(X) = \{t_x(\psi) : \psi \in X\}$.

Proof. Note that $\bigvee X$ exists in Φ_y , since $(\Phi_y; \leq)$ is a complete lattice. Consider a $\psi \in X$ so that $\psi \leq \bigvee X$, hence $t_x(\psi) \leq t_x(\bigvee X)$, thus $\bigvee t_x(X) \leq t_x(\bigvee X)$.

Conversely by density in Φ_x we have

$$t_x(\bigvee X) = \bigvee \{\psi \in \Phi_x : \psi \ll_x t_x(\bigvee X)\}.$$

By the compatibility condition, $\psi \ll_x t_x(\bigvee X)$ implies $t_y(\psi) \ll_y t_y(t_x(\bigvee X)) \leq \bigvee X$. By the definition of the way-below relation \ll_y this means that there is a $\chi \in X$ such that $t_y(\psi) \leq \chi$. But then it follows that $\psi = t_x(t_y(\psi)) \leq t_x(\chi) \in t_x(X)$, hence $t_x(\bigvee X) \leq \bigvee t_x(X)$ and therefore $t_x(\bigvee X) = \bigvee t_x(X)$. \square

Now we are in a position to prove the following theorem.

Theorem 7.21 *Let $(\Phi, D; \leq, \perp, \cdot, t)$ be a labeled continuous information algebra, and assume that D has a top element \top , then the associated dual domain-free information algebra $(\Phi/\sigma, D; \leq, \perp, \cdot, \epsilon)$ is continuous.*

Proof. We first show that $(\Phi/\sigma; \leq)$ is a complete lattice. To this end consider any non-empty subset $X \subseteq \Phi/\sigma$. For any element $[\psi]_\sigma$ of X we may take the representant ψ in the top domain Φ_\top , $d(\psi) = \top$. Let then $X' = \{\psi \in \Phi_\top : [\psi]_\sigma \in X\}$. But $(\Phi_\top; \leq)$ is a complete lattice, hence $\bigvee X'$ exists in Φ_\top . By Lemma 7.7, we have $[\bigvee X']_\sigma = \bigvee X$, and so X has a supremum in Φ/σ . Since $(\Phi/\sigma; \leq)$ has a smallest element $[1_\top]_\sigma$, by standard results of lattice theory $(\Phi/\sigma; \leq)$ is a complete lattice.

Next consider any class $[\phi]_\sigma \in \Phi/\sigma$. The set $\{[\psi]_\sigma : [\psi]_\sigma \ll [\phi]_\sigma\}$ is directed. Consider the representants of the classes of this set in Φ_\top : $\{\psi \in \Phi_\top : [\psi]_\sigma \ll [\phi]_\sigma\}$ and also $\phi \in \Phi_\top$. Then, by Lemma 7.7, Lemma 7.11 and density in the labeled algebra,

$$\begin{aligned} \bigvee \{[\psi]_\sigma : [\psi]_\sigma \ll [\phi]_\sigma\} &= [\bigvee \{\psi \in \Psi_\top : [\psi]_\sigma \ll [\phi]_\sigma\}]_\sigma \\ &= [\bigvee \{\psi \in \Psi_\top : \psi \ll_\top \phi\}]_\sigma = [\phi]_\sigma. \end{aligned}$$

This shows that (weak) density hold. Therefore, $(\Psi/\sigma; \leq)$ is a continuous lattice.

By Theorem 7.19 it is now sufficient to prove (7.28). So, consider a directed set $X \subseteq \Phi/\sigma$. For any $[\psi]_\sigma \in X$ we may select the representant ψ in Φ_\top . Define $X' = \{\psi \in \Phi_\top : [\psi]_\sigma \in X\}$. This set is still directed in Ψ_\top . Now, using repeatedly Lemma 7.7 and Lemma 7.12

$$\begin{aligned} \epsilon_x(\bigvee X) &= \epsilon_x(\bigvee \{[\phi]_\sigma : \phi \in X'\}) = \epsilon_x([\bigvee X']_\sigma) \\ &= [t_x(\bigvee X')]_\sigma = [\bigvee t_x(X')]_\sigma = \bigvee \{[t_x(\phi)]_\sigma : [\phi]_\sigma \in X\} \\ &= \bigvee \{\epsilon_x([\phi]_\sigma) : [\phi]_\sigma \in X\} = \bigvee \epsilon_x(X). \end{aligned}$$

This proves that $(\Psi/\sigma, D)$ is a domain-free continuous information algebra. \square

Note that the existence of a top element in $(D; \leq)$ is required in the proof above for $(\Psi/\sigma; \leq)$ to be continuous. We refer to the discussion at the end of Section ?? concerning the adjunction of a top domain to a continuous domain-free information algebra, hence the existence of this domain in the associated labeled algebra. This gives us the full duality between labeled and domain-free continuous information algebras. However, the definition of a continuous labeled information algebra makes also sense without a top domain. It remains so far an open question, whether a labeled continuous information algebra (Ψ, D) can be extended to a labeled continuous information algebra with a top domain. The problem is the extension of the compatibility condition to the new top domain.

As for compact information algebras, a way to obtain continuous information algebras is from continuous lattices as semiring information algebras, (Guang & Kohlas, 2015).

7.5 Atomic Algebras

In many idempotent information algebras there are maximal elements, called *atoms*. In important cases these maximal elements determine the algebra fully. For example, in the set algebra relative to a f.c.f $(\mathcal{F}, \mathcal{R})$, the elements or rather the one-element subsets of any frame, represent maximal information relative to the frame. We want to study this situation in the general framework of generalised idempotent information algebras. Labeled algebras are somewhat better suited for this subject than domain-free information algebras. However we refer to (Kohlas & Schmid, 2016) for a discussion of the same subject in the context of idempotent domain-free valuation algebras. In (Kohlas, 2003a) atoms in idempotent labeled valuation algebras were studied. This section presents a generalisation thereof.

Consider a labeled idempotent generalised information algebra $(\Psi, D; \leq, \perp, \cdot, t)$. Then we define the concept of an atom in this algebra as follows:

Definition 7.9 Atom: An element $\alpha \in \Psi_x$ in a domain $x \in D$ of a labeled idempotent generalised information algebra $(\Psi, D; \leq, \perp, \cdot, t)$ is called an atom in x , if

1. $\alpha \neq 0_x$,
2. for all $\psi \in \Psi_x$, $\alpha \leq \psi$ implies either $\alpha = \psi$ or $\psi = 0_x$.

So atoms are maximal elements in a domain, smaller than the null element¹. Since the null element does not represent proper information, atoms are indeed maximal pieces of information. We shall now first present a few elementary properties of atoms. Then we shall show that atoms are closely related to families of compatible frames (f.c.f). Finally, we present particular information algebras, which are essentially set algebras on some f.c.f.

Denote the set of atoms in domain x by At_x . Here are a few general properties of atoms:

Lemma 7.13 Let $(\Psi, D; \leq, \perp, \cdot, t)$ be an idempotent labeled generalised information algebra. Then the following holds:

1. If $y \leq x$, then $\alpha \in At_x$ and $\psi \in \Psi_y$ imply either $\psi \leq \alpha$ or else $\alpha \cdot \psi = 0_x$,
2. $\alpha, \beta \in At_x$ imply either $\alpha = \beta$ or else $\alpha \cdot \beta = 0_x$,
3. $\alpha \in At_x$ and $y \leq x$ imply $t_y(\alpha) \in At_y$,
4. $\alpha \in At_x$ and $\beta \in At_y$ imply either $\alpha \cdot \beta = 0_{x \vee y}$ or $t_x(\alpha \cdot \beta) = \alpha$ and $t_y(\alpha \cdot \beta) = \beta$,
5. $\alpha \in At_x$ and $\psi \in \Psi_y$ imply either $\alpha \cdot \psi = 0_{x \vee y}$ or $t_x(\alpha \cdot \psi) = \alpha$.

Proof. 1.) From $\alpha \leq \alpha \cdot \psi$ and the definition of an atom it follows that either $\alpha \cdot \psi = 0_x$ or $\alpha \cdot \psi = \alpha$, hence $\psi \leq \alpha$.

2.) As before, $\alpha, \beta \leq \alpha \cdot \beta$ implies either $\alpha \cdot \beta = 0_x$ or $\alpha \cdot \beta = \alpha$, $\alpha \cdot \beta = \beta$, hence $\alpha = \beta$.

3.) Since α is an atom, $\alpha \neq 0_x$ and therefore $t_y(\alpha) \neq 0_y$. Assume $t_y(\alpha) \leq \psi$, where $d(\psi) = y$. From $x \perp y | y$ we obtain $t_y(\alpha \cdot \psi) = t_y(\alpha) \cdot \psi = \psi$. Now, $\alpha \cdot \psi = \alpha$ or $\alpha \cdot \psi = 0_x$ by item 1 just proved. In the first case it follows that $t_y(\alpha) = \psi$, in the second case that $\psi = 0_y$. So, $t_y(\alpha)$ is indeed an atom.

4.) Assume $\alpha \cdot \beta \neq 0_{x \vee y}$. We have $x \perp y | x$, which implies $t_x(\alpha \cdot \beta) = \alpha \cdot t_x(\beta) \neq 0_x$ since otherwise $\alpha \cdot \beta = 0_{x \vee y}$. Then from $\alpha \leq \alpha \cdot t_x(\beta)$ it follows $\alpha = \alpha \cdot t_x(\beta)$. The second identity $t_y(\alpha \cdot \beta) = \beta$ follows in the same way.

¹In order theory atoms are usually defined as minimal elements. But for our purpose our definition fits better the idea of atoms as building blocks of a piece of information. See below.

5.) Is proved in the same way as item 4. \square

If for an element $\psi \in \Psi_x$ and an atom α we have $\psi \leq \alpha$, then α implies ψ . Therefore we define $At(\psi) = \{\alpha \in At_x : \psi \leq \alpha\}$ and call $At(\psi)$ the set of atoms implying ψ , or we say also that $At(\psi)$ is the set of atoms contained in ψ . Now, we fix our attention on information algebras, where every element contains an atom.

Definition 7.10 Atomic Information Algebra: A labeled idempotent generalised information algebra $(\Psi, D; \leq, \perp, \cdot, t)$ is called atomic, if for all $x \in D$ and all $\psi \in \Psi_x$, $\psi \neq 0_x$ implies $At(\psi) \neq \emptyset$.

In atomic information algebras, atoms have a few additional properties.

Lemma 7.14 Let $(\Psi, D; \leq, \perp, \cdot, t)$ be an atomic information algebra. Then the following holds:

1. If $x \leq y$, then for all $\alpha \in At_x$ there is an atom $\beta \in At_y$ such that $\alpha = t_x(\beta)$.
2. For all $\alpha \in At_x$, $\beta \in At_y$ such that $\alpha \cdot \beta \neq 0_{x \vee y}$, there is an atom $\gamma \in At_{x \vee y}$ such that $t_x(\gamma) = \alpha$ and $t_y(\gamma) = \beta$.

Proof. 1.) Since the algebra is atomic, there exists an atom $\beta \in At(t_y(\alpha))$, $t_y(\alpha) \neq 0_y$. Then $t_y(\alpha) \leq \beta$ which implies $\alpha = t_x(t_y(\alpha)) \leq t_x(\beta)$. But $t_x(\beta) \neq 0_x$ since $\beta \neq 0_y$, hence $\alpha = t_x(\beta)$.

2.) Again there is an atom $\gamma \in At(\alpha \cdot \beta)$, such that $\alpha \cdot \beta \leq \gamma$. But then by Lemma 7.13, item 4, $\alpha = t_x(\alpha \cdot \beta) \leq t_x(\gamma) \neq 0_x$. So $\alpha = t_x(\gamma)$. Similarly, we derive $\beta = t_y(\gamma)$. \square

Now, we consider the sets At_x and show that they are part of a compatible family of frames (f.c.f), if the underlying algebra is atomic. We start by showing how the transport operation induces refinings among sets of atoms At_x . Define for $x \leq y$ and $\alpha \in At_x$ the map

$$\tau_{x,y}(\alpha) = At(t_y(\alpha)) \quad (7.30)$$

from At_x into the power set of At_y .

Theorem 7.22 Let $(\Psi, D; \leq, \perp, \cdot, t)$ be an atomic information algebra. Then $\tau_{x,y}$ is a refining of At_x .

Proof. First, $\tau_{x,y}(\alpha) \neq \emptyset$, since the algebra is atomic. Secondly, consider atoms $\alpha \neq \beta$. Assume that $\tau_{x,y}(\alpha) \cap \tau_{x,y}(\beta) \neq \emptyset$. Then select an atom γ in $\tau_{x,y}(\alpha) \cap \tau_{x,y}(\beta)$. But this means $\gamma \in At(t_y(\alpha))$, hence $t_y(\alpha) \leq \gamma$. It follows that $\alpha = t_x(t_y(\alpha)) \leq t_x(\gamma)$. But this implies $t_x(\gamma) = \alpha$, since $t_x(\gamma)$ is atom in x (Lemma 7.13). In the same way we deduce $t_x(\gamma) = \beta$, hence $\alpha = \beta$, contrary to the assumption. So $\tau_{x,y}(\alpha) \cap \tau_{x,y}(\beta) = \emptyset$. Finally consider any

atom γ in At_y . Since $\alpha = t_x(\gamma)$ is an atom in x , we conclude $\gamma \in \tau_{x,y}$. This shows that $\cup_{\alpha \in At_x} \tau_{x,y}(\alpha) = At_y$ and this concludes the proof that $\tau_{x,y}$ is a refining of x . \square

According to this theorem, in an atomic information algebra, if $x \leq y$, then At_y is a refinement of At_x and the latter a coarsening of At_y . We may extend the maps $\tau_{x,y}$ in the usual way to sets of atoms in x . Let now $\mathcal{F} = \{At_x : x \in D\}$ and $\mathcal{R} = \{\tau_{x,y} : x, y \in D, x \leq y\}$. We claim that $(\mathcal{F}, \mathcal{R})$ is a f.c.f, provided that some additional conditions are satisfied. This issue will be addressed below.

Consider now first the family of all sets of atoms $At(\psi)$ for $\psi \in \Psi$. Let's denote this family by $\mathcal{S}_{At(\Psi)}$. We define the following operations with respect to these sets of atoms:

1. *Labeling*: $d(At(\psi)) = At_x$ if $d(\psi) = x$.

2. *Combination*: If $d(\phi) = x$ and $d(\psi) = y$,

$$At(\phi) \bowtie At(\psi) = \{\gamma \in At_{x \vee y} : t_x(\gamma) \in At(\phi), t_y(\gamma) \in At(\psi)\}. \quad (7.31)$$

3. *Transport*: If $d(\psi) = x$ and $y \in D$,

$$t_y(At(\psi)) = v_{y, x \vee y}(\tau_{x, x \vee y}(At(\psi))). \quad (7.32)$$

Here $v_{y, x \vee y}(A)$ is the saturation operator (see Section 2.3), defined by $v_{y, x \vee y}(A) = \{\alpha \in At_y : \tau_{y, x \vee y}(\alpha) \cap A \neq \emptyset\}$ for $A \subseteq At_{x \vee y}$. Note also the similarity of combination with the relational join in relational algebra. We show that $At(\phi) \bowtie At(\psi)$ and $t_x(At(\psi))$ are elements of $\mathcal{S}_{At(\Psi)}$. This follows from the following theorem:

Theorem 7.23 *Let $(\Psi, D; \leq, \perp, \cdot, t)$ be an atomic information algebra. Then for all $\phi, \psi \in \Psi$ and $y \in D$,*

$$\begin{aligned} At(\phi) \bowtie At(\psi) &= At(\phi \cdot \psi) \\ t_y(At(\psi)) &= At(t_y(\psi)). \end{aligned}$$

Proof. Consider first $\alpha \in At(\phi \cdot \psi)$. Then $\phi \cdot \psi \leq \alpha$. Assume $d(\phi) = x$ and $d(\psi) = y$. From $x \perp y | x$ it follows that $\phi \leq \phi \cdot t_x(\psi) = t_x(\phi \cdot \psi) \leq t_x(\alpha) \in At_x$. This shows that $t_x(\alpha) \in At(\phi)$. Similarly it follows that $t_y(\alpha) \in At(\psi)$. So we conclude that $\alpha \in At(\phi) \bowtie At(\psi)$ and $At(\phi \cdot \psi) \subseteq At(\phi) \bowtie At(\psi)$. Conversely, if $\alpha \in At(\phi) \bowtie At(\psi)$, then $\alpha \in At_{x \vee y}$ and $t_x(\alpha) \in At(\phi)$, $t_y(\alpha) \in At(\psi)$. It follows that $\phi \cdot \psi \leq t_x(\alpha) \cdot t_y(\alpha) \leq \alpha$ and thus $\alpha \in At(\phi \cdot \psi)$. This proves that, $At(\phi) \bowtie At(\psi) = At(\phi \cdot \psi)$.

Next consider an atom $\alpha \in At(t_y(\psi))$ and assume $d(\psi) = x$. In order to show that $\alpha \in t_y(At(\psi))$ we need to verify that

$$\tau_{y, x \vee y}(\alpha) \cap \tau_{x, x \vee y}(At(\psi)) \neq \emptyset. \quad (7.33)$$

Note that $t_y(\alpha \cdot \psi) = \alpha \cdot t_y(\psi) = \alpha \neq 0_y$, hence $\alpha \cdot \psi \neq 0_{x \vee y}$. Therefore there exists an atom β in $At(\alpha \cdot \psi)$, such that $\alpha \cdot \psi \leq \beta$. Then $t_x(\beta) \geq t_x(\alpha \cdot \psi) = t_x(\alpha) \cdot \psi \geq \psi$ which shows that $t_x(\beta) \in At(\psi)$, hence $\beta \in \tau_{x, x \vee y}(At(\psi))$. But we have also $t_y(\beta) \geq t_y(\alpha \cdot \psi) = \alpha$, thus $t_y(\beta) = \alpha$, which shows that $\beta \in \tau_{y, x \vee y}(\alpha)$. Thus (7.33) holds and $\alpha \in t_y(At(\psi))$.

Conversely, consider $\alpha \in t_y(At(\psi))$. Then α is an atom in y such that (7.33) holds. Select then an atom β in this intersection. We have $t_y(\beta) = \alpha$ and $t_x(\beta) \in At(\psi)$ or $t_x(\beta) \geq \psi$. This implies $\beta \geq t_{x \vee y}(t_x(\beta)) \geq t_{x \vee y}(\psi)$. Then $\alpha = t_y(\beta) \geq t_y(t_{x \vee y}(\psi)) = t_y(\psi)$. So we conclude that $\alpha \in At(t_y(\psi))$ and thus $t_y(At(\psi)) = At(t_y(\psi))$. \square

According to this theorem $\mathcal{S}_{At(\Psi)}$ is closed under the operation of combination and transport defined above. As in Section 3.1, where we introduced a set algebra as a labeled idempotent information algebra relative to a f.c.f. $(\mathcal{F}, \mathcal{R})$, we consider pairs $(At(\psi), At_x)$ for $d(\psi) = x$ and define $\Psi_{At_x} = \{(At(\psi), At_x) : d(\psi) = x\}$ and $\Psi_{At(\Psi)} = \cup_{x \in D} \Psi_{At_x}$. The operations above in $\mathcal{S}_{At(\Psi)}$ can then be extended to Ψ in the following way:

1. *Labeling:* $d(At(\psi), At_x) = At_x$.
2. *Combination:* $(At(\phi), At_x) \cdot (At(\psi), At_y) = (At(\phi) \bowtie At(\psi), At_x \vee At_y)$.
3. *Transport:* $t_{At_y}(At(\psi), At_x) = (t_y(At(\psi)), At_y)$,

Note that $(At(1_x), At_x) = (At_x, At_x)$ and $(At(0_x), At_x) = (\emptyset, At_x)$ are the unit and null elements of combination in domain At_x . The map $\psi \mapsto (At(\psi), At_x)$ for $d(\psi) = x$ satisfies, according to Theorem 7.23,

$$\begin{aligned} \phi \cdot \psi &\mapsto (At(\phi), At_x) \cdot (At(\psi), At_y), \\ 1_x &\mapsto (At_x, At_x), \\ 0_x &\mapsto (\emptyset, At_x), \\ t_y(\psi) &\mapsto t_{At_y}(At(\psi), x). \end{aligned} \tag{7.34}$$

All this indicates that the algebra of subsets of atoms in $\mathcal{S}_{At(\Psi)}$ might be a generalised information algebra, somehow connected to the original atomic algebra $(\Psi, D; \leq, \perp, \cdot, t)$. To pursue this line of inquiry we strengthen the concept of an atomic algebra

Definition 7.11 Atomistic Information Algebras: A labeled idempotent generalised information algebra $(\Psi, D; \leq, \perp, \cdot, t)$ is called atomistic, if it is atomic and if for all $\psi \in \Psi_x$, $\psi \neq 0_x$,

$$\psi = \bigwedge At(\psi). \tag{7.35}$$

The family of sets At_x for $x \in D$ is nearly a family of compatible frames. Only the Identity of Coarsenings condition does not hold in general ². We introduce an additional condition, which proves to be necessary and sufficient to guarantee the Identity of Coarsenings between the frames At_x . We call two domains x and y from D *informorph* if

$$\begin{aligned} &\text{for all } \alpha \in At_x \text{ there exists } \beta \in At_y \\ &\text{and for all } \beta \in At_y \text{ there exists } \alpha \in At_x \\ &\text{such that } \alpha \equiv_\sigma \beta. \end{aligned} \tag{7.36}$$

This means essentially that atoms in domains x and y convey the same information. In fact, if (7.36) holds, then $t_{x \vee y}(\alpha) = t_{x \vee y}(\beta)$, hence $t_y(\alpha) = t_y(t_{x \vee y}(\alpha)) = t_y(t_{x \vee y}(\beta)) = \beta$ and similarly $t_x(\beta) = \alpha$. So, we have $t_x(t_y(\alpha)) = \alpha$ and also $t_y(t_x(\beta)) = \beta$ for all atoms in At_x or At_y respectively if domains x and y are infomorph.

Define now $\mathcal{F} = \{At_x : x \in D\}$ and $\mathcal{R} = \{\tau_{x,y} : x \leq y\}$. Then $(\mathcal{F}, \mathcal{R})$ is an f.c.f provided that no two domains x and y in D are informorph.

Theorem 7.24 *Let $(\Psi, D; \leq, \perp, d, \cdot, t)$ be an atomistic information algebra such that no two different domains are infomorphic. Then $(\mathcal{F}, \mathcal{R})$ is a family of compatible frames (f.c.f).*

Proof. By definition we have for $x \leq y \leq z$,

$$\begin{aligned} \tau_{y,z} \circ \tau_{x,y}(\alpha) &= At(t_z(At(t_y(\alpha)))) \\ &= \{\gamma \in At(t_z(\beta)) \text{ for some } \beta \in At(t_y(\alpha))\}. \end{aligned}$$

This means $\gamma \geq t_z(\beta)$ and $\beta \geq t_y(\alpha)$, hence $\gamma \geq t_z(t_y(\alpha))$. Since $y \perp z | y$ and $x \leq y$ we have also $x \perp z | y$ and therefore $t_z(t_y(\alpha)) = t_z(\alpha)$. Therefore, $\gamma \geq t_z(\alpha)$, that is $\gamma \in At(t_z(\alpha))$. So we conclude that $\tau_{y,z} \circ \tau_{x,y}(\alpha) = At(t_z(\alpha)) = \tau_{x,z}(\alpha)$. Therefore, $\tau_{y,z} \circ \tau_{x,y} = \tau_{x,z} \in \mathcal{R}$. This shows that composition of refinings holds.

For all $\alpha \in At_x$ we have $t_x(\alpha) = \alpha$. So $\tau_{x,x}$ is the identity map $\tau_{x,x}(\alpha) = \{\alpha\}$. This is the identity condition. The identity of refining is obvious. Clearly $At_{x \vee y}$ is the minimal common refinement of At_x and At_y and $\tau_{x,x \vee y}$ and $\tau_{y,x \vee y}$ are the corresponding refinings and for all atoms $\alpha \in At_{x \vee y}$ we have $\tau_{x,x \vee y}(\beta) \cap \tau_{y,x \vee y}(\gamma) = \{\alpha\}$ if $\beta = t_x(\alpha)$ and $\gamma = t_y(\alpha)$.

So far, the assumption that no two domains are informorphic is not used. This assumption means that if (7.36) holds for a pair of domains $x, y \in D$, then $x = y$. Consider then domains $x, y, z \in D$ where $x, y \leq z$, such that for each atom α in x there is an atom β in domain y such that $\tau_{x,z}(\alpha) = \tau_{y,z}(\beta)$.

²This means that the frames At_x form only a preorder under the order induced by refinings. This is may be an indication that a preorder of domains or frames could be sufficient to develop the theory of generalised information algebra, see (Dawid, 2001).

This means that $At(t_z(\alpha)) = At(t_z(\beta))$ or that $t_z(\alpha) = t_z(\beta)$ since the algebra is assumed to be atomistic. But then $t_{x \vee y}(\alpha) = t_{x \vee y}(t_z(\alpha)) = t_{x \vee y}(t_z(\beta)) = t_{x \vee y}(\beta)$. So, we have $\alpha \equiv_\sigma \beta$. In the same way we find for each atom β in domain y an atom α in domain x such that $\alpha \equiv_\sigma \beta$. But then the assumption implies $x = y$, hence $At_x = At_y$. So Identity of Coarsening holds in $(\mathcal{F}, \mathcal{R})$. \square

In this f.c.f $(\mathcal{F}, \mathcal{R})$ we may of course define the relation of conditional independence between frames as in Section 2.3. Provided the conditional independence relation defines a q-separoid, we may, as in Section 3.1, define the generalised information algebra $(\Phi, \mathcal{F}; \leq, \perp, d, \cdot, t)$ of subsets of atoms if the original algebra $(\Psi, D; \leq, \perp, d, \cdot, t)$ is atomistic. We shall see below that the original algebra is in fact embedded into this algebra of subsets of atoms.

But before we turn to this issue, we remark that the conditional independence relation in the f.c.f $(\mathcal{F}, \mathcal{R})$ is closely related to the relation $x \perp y | z$ in the q-separoid $(D; \leq, \perp)$. First of all, we have that $x \leq y$ in D implies $At_x \leq At_y$ in $(\mathcal{F}; \leq, \perp)$ (we make as usual no distinction in the notation of relations in both structure $(D; \leq, \perp)$ and $(\mathcal{F}; \leq, \perp)$). Further, we have $At_x \vee At_y = At_{x \vee y}$. So, as partial orders $(\mathcal{F}; \leq, \perp)$ is homomorphic to $(D; \leq, \perp)$. In fact, there is more as the following theorem shows.

Theorem 7.25 *Let $(\Psi, D; \leq, \perp, \cdot, t)$ be an atomic information algebra. Then $x \perp y | z$ implies $At_x \perp At_y | At_z$.*

Proof. We start by recalling the definition of the relation $At_x \perp At_y | At_z$ in $(\mathcal{F}, \mathcal{R})$. For $\alpha \in At_x$, $\beta \in At_y$ and $\gamma \in At_z$ we have (Section 2.3),

$$R_\gamma(At_x, At_y) = \{(\alpha, \beta) : (\alpha, \beta, \gamma) \in R(At_x, At_y, At_z)\}.$$

Further, $(\alpha, \beta, \gamma) \in R(At_x, At_y, At_z)$ means that

$$\tau_{x, x \vee y \vee z}(\alpha) \cap \tau_{y, x \vee y \vee z}(\beta) \cap \tau_{z, x \vee y \vee z}(\gamma) \neq \emptyset. \quad (7.37)$$

Similarly,

$$\begin{aligned} R_\gamma(At_x) &= \{\alpha : (\alpha, \gamma) \in R(At_x, At_z)\}, \\ R_\gamma(At_y) &= \{\beta : (\beta, \gamma) \in R(At_y, At_z)\}. \end{aligned}$$

Here we have $(\alpha, \gamma) \in R(At_x, At_z)$ and $(\beta, \gamma) \in R(At_y, At_z)$ if

$$\tau_{x, x \vee z}(\alpha) \cap \tau_{z, x \vee z}(\gamma) \neq \emptyset \text{ and } \tau_{y, y \vee z}(\beta) \cap \tau_{z, y \vee z}(\gamma) \neq \emptyset. \quad (7.38)$$

Finally, $At_x \perp At_y | At_z$ means that

$$R_\gamma(At_x, At_y) = R_\gamma(At_x) \times R_\gamma(At_y) \quad (7.39)$$

for all $\gamma \in At_z$.

Remember that we have always $R_\gamma(At_x, At_y) \subseteq R_\gamma(At_x) \times R_\gamma(At_y)$. Assume then (7.38). Remind that $x \perp y | z$ implies $x \vee z \perp y \vee z | z$. Since we assume $\tau_{x, x \vee z}(\alpha) \cap \tau_{z, x \vee z}(\gamma) \neq \emptyset$ there is an atom α' in the intersection $At(t_{x \vee z}(\alpha)) \cap At(t_{x \vee z}(\gamma))$. Then $t_{x \vee z}(\alpha), t_{x \vee z}(\gamma) \leq \alpha'$ so that $\alpha = t_x(t_{x \vee z}(\alpha)) \leq t_x(\alpha')$. Since $t_x(\alpha')$ is an atom in x , we conclude that $\alpha = t_x(\alpha')$. In the same way we obtain also $\gamma = t_z(\alpha')$. Similarly select an atom β' in $At(t_{y \vee z}(\beta)) \cap At(t_{y \vee z}(\gamma))$ and conclude in the same way that $\beta = t_y(\beta')$ and $\gamma = t_z(\beta')$.

Since $x \vee z \perp y \vee z | z$ we obtain $t_z(\alpha' \cdot \beta') = t_z(\alpha') \cdot t_z(\beta') = \gamma$. Since $\gamma \neq 0_z$ we infer that $\alpha' \cdot \beta' \neq 0_{x \vee y \vee z}$. Therefore there is an atom $\gamma' \in At(\alpha' \cdot \beta')$, that is, $\gamma' \geq \alpha' \cdot \beta'$. We have

$$\alpha' \cdot \beta' = t_{x \vee y \vee z}(\alpha') \cdot t_{x \vee y \vee z}(\beta') \geq t_{x \vee y \vee z}(\alpha') \geq t_{x \vee y \vee z}(t_{x \vee z}(\alpha)) = t_{x \vee y \vee z}(\alpha).$$

So, γ' is an atom in $At(t_{x \vee y \vee z}(\alpha))$. In the same way we obtain that $\gamma' \geq t_{x \vee y \vee z}(\beta)$, so that γ' is also an element of $At(t_{x \vee y \vee z}(\beta))$. Finally $\gamma' \geq \alpha' \cdot \beta'$ implies $\gamma' \geq t_z(\alpha' \cdot \beta') = t_z(\alpha') \cdot t_z(\beta') = \gamma$. From this we conclude also that $\gamma' \geq t_{x \vee y \vee z}(\gamma)$, hence γ' belongs also to $At(t_{x \vee y \vee z}(\gamma))$. But this shows that (7.37) holds. And this concludes the proof. \square

Remark that in the proof of this theorem, no use is made of the assumption that $(\mathcal{F}, \mathcal{R})$ is an f.c.f, that is that no two different domains are infomorph. But we now make again this assumption such that $(\mathcal{F}, \mathcal{R})$ becomes a f.c.f and the algebra of subsets of atoms $(\Phi, \mathcal{F}; \leq, \perp, d, \cdot, t)$ a generalised, idempotent information algebra and turn to the question how the original atomistic algebra $(\Psi, D; \leq, \perp, d, \cdot, t)$ is related to this algebra. We have introduced above $\Psi_{At(\Psi)}$ as the set of pairs $(At(\psi), At_x)$, if $d(\psi) = x$. They belong to the set Φ . So, the map $\psi \mapsto (At(\psi), At_x)$ maps Ψ into Φ .

In summary, we have the following maps between $(\Psi, D; \leq, \perp, d, \cdot, t)$ and $(\Phi, \mathcal{F}; \leq, \perp, d, \cdot, t)$

$$\begin{aligned} \psi \in \Psi &\mapsto (At(\psi), At_x) \in \Phi \text{ if } d(\psi) = x, \\ x \in D &\mapsto At_x \in \mathcal{F}, \\ t_x : \Psi \rightarrow \Psi &\mapsto t_{At_x} : \Phi \rightarrow \Phi. \end{aligned}$$

If $(\Psi, D; \leq, \perp, d, \cdot, t)$ is atomistic and no two domains are infomorph, then these maps satisfy the homomorphy conditions (7.34) and further the map $x \mapsto At_x$ maintains order and $x \perp y | z$ implies $At_x \perp At_y | At_z$ (Theorem 7.25), is therefore a q-separoid homomorphism (Dawid, 2001). Further, $At(\phi) = At(\psi)$ implies $\phi = \wedge At(\phi) = \wedge At(\psi) = \psi$ and $At_x = At_y$ implies $x = y$. So these maps are one-to-one. Therefore, we may speak of an embedding of the atomistic information algebra $(\Psi, D; \leq, \perp, d, \cdot, t)$ into the generalised information algebra $(\Phi, \mathcal{F}; \leq, \perp, d, \cdot, t)$ of sets of atoms. In other words, an atomistic information algebra may be represented by the information algebra of the sets of its atoms. Each piece of information ψ is represented by the set

of its atoms $At(\psi)$. We may consider the atoms in $At(\psi)$ the set of possible answers compatible with the piece of information ψ . In this way any piece of information is represented by its set of possible answers. A more complete study of representation of idempotent valuation algebras by set algebras is given in (Kohlas & Schmid, 2016).

Part III

Constructing New Algebras

Chapter 8

Information Maps

8.1 Continuous Maps

There are many ways to construct new information or valuation algebras from old ones. For instance, maps from any set into a generalised information algebra or a valuation algebra form again an information or valuation algebra under point-wise combination and extraction. In this section however, we consider order-preserving maps, between idempotent domain-free valuation algebras and show that these structures form Cartesian closed categories. It turns out, that only order-preserving maps between idempotent *valuation algebras*, rather than idempotent generalised information algebras lead to a satisfactory theory of Cartesian closed categories.

Consider two idempotent, domain-free information algebras $(\Psi_1, E_1; \cdot, \circ)$ and $(\Psi_2, E_2; \cdot, \circ)$ satisfying each axioms D0 to D5 of Section 5.2. A map $f : \Psi_1 \rightarrow \Psi_2$ is order-preserving, if $\phi_1 \leq \psi_1$ in Ψ_1 implies $f(\phi_1) \leq f(\psi_1)$ in Ψ_2 , a more informative piece of information is mapped to a more informative piece of information. For the maps to be considered here, we require more: The null element in Ψ_1 , and only the null element, should map to the null element in Ψ_2 , the map f can neither eliminate nor create contradiction. This leads us to the following definition:

Definition 8.1 *If $(\Psi_1, E_1; \cdot, \circ)$ and $(\Psi_2, E_2; \cdot, \circ)$ are two domain-free valuation algebras satisfying axioms D0 to D5 (Section 5.2), then an order-preserving map $f : \Psi_1 \rightarrow \Psi_2$ such that $f(\psi) = 0$ if and only if $\psi = 0$ is called an information map. If furthermore $f(1) = 1$, the information map is called strict.*

In this definition, as well as in the sequel it should be clear that the symbols 0 and 1 denote unit and null elements both in Ψ_1 and Ψ_2 according to the context, we do not differentiate between them by notation. The same holds for operations and relational symbols, it will always be clear from the context, whether the operation or relation is in Ψ_1 or Ψ_2 .

Denote the set of all information maps between Ψ_1 and Ψ_2 by $[\Psi_1 \rightarrow \Psi_2]$. We define the following operations for information maps $f, g \in [\Psi_1 \rightarrow \Psi_2]$ and extraction operators $\epsilon_1 \in E_1$ and $\epsilon_2 \in E_2$:

1. *Combination*: $f \cdot g$ defined by $(f \cdot g)(\psi) = f(\psi) \cdot g(\psi)$ for all $\psi \in \Psi_1$,
2. *Extraction*: $(\epsilon_1, \epsilon_2)(f)$ defined by $(\epsilon_1, \epsilon_2)(f)(\psi) = \epsilon_2(f(\epsilon_1(\psi)))$ for all $\psi \in \Psi_1$.

It is obvious that $f \cdot g, (\epsilon_1, \epsilon_2)(f) \in [\Psi_1 \rightarrow \Psi_2]$, so $[\Psi_1 \rightarrow \Psi_2]$ is closed both under combination as well as extraction. The Cartesian product $E_1 \times E_2$ is a semigroup under coordinate-wise composition, $(\epsilon_1, \epsilon_2) \circ (\eta_1, \eta_2) = (\epsilon_1 \circ \eta_1, \epsilon_2 \circ \eta_2)$. In fact, we show that these operations define an idempotent domain-free valuation algebra of information maps.

Theorem 8.1 *If $(\Psi_1, E_1; \cdot, \circ)$ and $(\Psi_2, E_2; \cdot, \circ)$ are two domain-free valuation algebras satisfying axioms D0, D1 and D3, D4, D5 (Section 5.2), then $([\Psi_1 \rightarrow \Psi_2], E_1 \times E_2; \cdot, \circ)$ satisfies the same axioms.*

Proof. To verify axiom D0, consider the composition of (ϵ_1, ϵ_2) and (η_1, η_2) ,

$$\begin{aligned} & ((\epsilon_1, \epsilon_2) \circ (\eta_1, \eta_2))(f)(\psi) \\ &= (\epsilon_1, \epsilon_2)((\eta_1, \eta_2)(f))(\psi) = \epsilon_2(\eta_2(f(\eta_1(\epsilon_1(\psi))))) \end{aligned}$$

Commutativity and idempotency of composition in $E_1 \times E_2$ follows then from commutativity and idempotency of composition in E_1 and E_2 . Therefore, axiom D0 holds. It is clear that information maps form an idempotent, commutative semigroup under combination. The unit and null elements in $[\Psi_1 \rightarrow \Psi_2]$ are the maps 1 and 0 defined by $1(\psi) = 1$ and $0(\psi) = 0$ for all $\psi \in \Psi_1$. So axiom D1 is satisfied. Further, we have $(\epsilon_1, \epsilon_2)(1)(\psi) = \epsilon_2(1(\epsilon_1(\psi))) = 1$ and similarly $(\epsilon_1, \epsilon_2)(0)(\psi) = \epsilon_2(0(\epsilon_1(\psi))) = 0$. So, $(\epsilon_1, \epsilon_2)(1) = 1$ and $(\epsilon_1, \epsilon_2)(0) = 0$. Further, if $(\epsilon_1, \epsilon_2)(f)(\psi) = \epsilon_2(f(\epsilon_1(\psi))) = 0$ then by axiom D3 in Ψ_1 and Ψ_2 we have $f(\epsilon_1(\psi)) = 0$, hence $\epsilon_1(\psi) = 0$, therefore $\psi = 0$. So, $(\epsilon_1, \epsilon_2)(f) = 0$ implies $f = 0$. This shows that axiom D3 is satisfied for information maps. Then, using axiom D4 in Ψ_2 , we have for any $\psi \in \Psi_1$,

$$\begin{aligned} & (\epsilon_1, \epsilon_2)((\epsilon_1, \epsilon_2)(f) \cdot g)(\psi) \\ &= \epsilon_2((\epsilon_1, \epsilon_2)(f) \cdot g)(\epsilon_1(\psi)) = \epsilon_2((\epsilon_1, \epsilon_2)(f)(\epsilon_1(\psi)) \cdot g(\epsilon_1(\psi))) \\ &= \epsilon_2(\epsilon_2(f(\epsilon_1(\epsilon_1(\psi)))) \cdot g(\epsilon_1(\psi))) = \epsilon_2(f(\epsilon_1(\psi)) \cdot \epsilon_2(g(\epsilon_1(\psi)))) \\ &= ((\epsilon_1, \epsilon_2)(f) \cdot (\epsilon_1, \epsilon_2)(g))(\psi). \end{aligned}$$

So, we see that $(\epsilon_1, \epsilon_2)((\epsilon_1, \epsilon_2)(f) \cdot g) = (\epsilon_1, \epsilon_2)(f) \cdot (\epsilon_1, \epsilon_2)(g)$ and this is axiom D4. Finally, axiom D5 follows from

$$(f \cdot (\epsilon_1, \epsilon_2)(f))(\psi) = f(\psi) \cdot \epsilon_2(f(\epsilon_1(\psi))) = f(\psi)$$

since $\epsilon_1(\psi) \leq \psi$, hence $f(\epsilon_1(\psi)) \leq f(\psi)$. \square

The Support Axiom D2 is, in general, not satisfied by information maps, even if it is satisfied in Ψ_1 and Ψ_2 . We may trivially always adjoin the identity map id both to E_1 and E_2 , then axiom D2 is trivially satisfied in Ψ_1 and Ψ_2 , even if it was not before, and it is then also satisfied in $[\Psi_1 \rightarrow \Psi_2]$. Note that this axiom is only important for duality between domain-free and labeled algebras. For valuation algebras the duality plays only, when the extraction operators form a lattice, so duality is in this framework of less interest, and the support axiom is not needed.

Next, we consider *continuous* valuation algebras. In section 7.4 we have introduced and discussed continuous generalised information algebras. Continuous valuation algebras are defined analogously and the same result hold for valuation algebras as for generalised information algebras, see (Kohlas, 2003a; Kohlas & Schmid, 2014). In this context, we consider *continuous* maps:

Definition 8.2 *If $(\Psi_1, E_1; \cdot, \circ)$ and $(\Psi_2, E_2; \cdot, \circ)$ are two continuous domain-free valuation algebras with bases B_1 and B_2 respectively, then a map $f : \Psi_1 \rightarrow \Psi_2$ is called continuous, if for all $\psi \in \Psi_1$,*

$$f(\psi) = \bigvee \{f(\phi) : \phi \in B_1, \phi \ll \psi\}$$

and $f(\psi) = 0$ if and only if $\psi = 0$.

Let $[\Psi_1 \rightarrow \Psi_2]_c$ denote the set of continuous maps between Ψ_1 and Ψ_2 . A continuous map is obviously order-preserving, hence an information map.

Continuity of maps is a purely order-theoretic concept and there are several equivalent definitions (Davey & Priestley, 2002):

Lemma 8.1 *The following are equivalent:*

1. $f(\psi) = \bigvee \{f(\phi) : \phi \in B_1, \phi \ll \psi\}$ for all $\psi \in \Psi_1$,
2. $\{\phi \in B_2 : \phi \ll f(\psi)\} \subseteq \{\phi \in \Psi_2 : \phi \leq f(\chi), \chi \ll \psi \text{ for some } \chi \in B_1\}$ for all $\psi \in \Psi_1$,
3. if $X \subseteq \Psi_1$ is directed, then

$$f(\bigvee X) = \bigvee_{\psi \in X} f(\psi).$$

Proof. (1) \Rightarrow (2) : Consider an element $\phi \in B_2$ such that $\phi \ll f(\psi)$. Then we have by (1)

$$\phi \ll f(\psi) = \bigvee \{f(\chi) : \chi \in B_1, \chi \ll \psi\}.$$

The set $\{f(\chi) : \chi \in B_1, \chi \ll \psi\}$ is directed. Therefore, there is an element $\chi \in B_1$ with $\chi \ll \psi$, and such that $\phi \leq f(\chi)$. So (2) holds.

(2) \Rightarrow (3) : Consider a directed subset X of Ψ_1 and define $\psi = \bigvee X$. If $\phi \in B_2$ such that $\phi \ll f(\psi)$, then there exists by (2) an element $\chi \in B_1$ such that $\chi \ll \psi$ and $\phi \leq f(\chi)$. There is then further an element $\eta \in X$ such that $\chi \leq \eta$. Hence we conclude that $\phi \leq f(\chi) \leq f(\eta) \leq \bigvee f(X)$. So, by continuity in Ψ_2 , we have

$$f(\bigvee X) = \bigvee \{\phi \in B_2 : \phi \ll f(\bigvee X)\} \leq \bigvee f(X).$$

Obviously, $f(\bigvee X) \geq \bigvee f(X)$, so that $f(\bigvee X) = \bigvee f(X)$, hence (3) holds.

(3) \Rightarrow (1) : By continuity in Ψ_1 , we have $\psi = \bigvee \{\phi \in B_1 : \phi \ll \psi\}$, the set $\{\phi \in B_1 : \phi \ll \psi\}$ is directed, and therefore, (1) follows from (3), \square

As a corollary, it follows from Theorem 7.19 that the extraction operators ϵ of a continuous idempotent valuation algebra $(\Psi, E; \cdot, \circ)$ are continuous maps, hence belongs to $[\Psi \rightarrow \Psi]_c$. We proceed to show that combination and extraction operators of continuous maps produce continuous maps. This implies then, that $([\Psi_1 \rightarrow \Psi_2]_c, E_1 \times E_2; \cdot, \circ)$ is again an idempotent continuous valuation algebra, in fact, a subalgebra of $([\Psi_1 \rightarrow \Psi_2], E_1 \times E_2; \cdot, \circ)$.

Theorem 8.2 *If $(\Psi_1, E_1; \cdot, \circ)$ and $(\Psi_2, E_2; \cdot, \circ)$ are two continuous domain-free valuation algebras, $f, g \in [\Psi_1 \rightarrow \Psi_2]_c$, $(\epsilon_1, \epsilon_2) \in E_1 \times E_2$, then $f \cdot g, (\epsilon_1, \epsilon_2)(f) \in [\Psi_1 \rightarrow \Psi_2]_c$.*

Proof. The proof is straightforward using item 3 of Lemma 8.1 and continuity of extractor operators in Ψ_1 and Ψ_2 . So, let X be a directed subset of Ψ_1 , then

$$\begin{aligned} (f \cdot g)(\bigvee X) &= f(\bigvee X) \vee g(\bigvee X) = (\bigvee_{\psi \in X} f(\psi)) \vee (\bigvee_{\psi \in X} g(\psi)) \\ &= \bigvee_{\psi \in X} (f(\psi) \vee g(\psi)) = \bigvee_{\psi \in X} (f \cdot g)(\psi). \end{aligned}$$

This shows that $f \cdot g$ is continuous.

In a similar way, since both $\epsilon_1(X)$ and $f(\epsilon_1(X))$ are directed sets,

$$\begin{aligned} (\epsilon_1, \epsilon_2)(f)(\bigvee X) &= \epsilon_2(f(\epsilon_1(\bigvee X))) = \epsilon_2(f(\bigvee_{\psi \in X} \epsilon_1(\psi))) \\ &= \bigvee_{\psi \in X} \epsilon_2(f(\epsilon_1(\psi))) = \bigvee_{\psi \in X} (\epsilon_1, \epsilon_2)(f)(\psi). \end{aligned}$$

This shows that $(\epsilon_1, \epsilon_2)(f)$ is a continuous map. \square

It is well-known from order theory that $([\Psi_1 \rightarrow \Psi_2]_c; \leq)$ is a continuous lattice. Then we can use Theorem 7.19 to show that $([\Psi_1 \rightarrow \Psi_2]_c, E_1 \times$

$E_2; \cdot, \circ$) is a continuous valuation algebra. Before we do that (Theorem 8.4 below), we want to be more explicite and introduce a natural basis in $[\Psi_1 \rightarrow \Psi_2]_c$. Let Y be a finite subset of the basis B_1 of Ψ_1 . We call a function $s : Y \rightarrow B_2$ a *simple function*. Let S be the set of simple function. For any simple function s let $Y(s)$ denote the domain of s and define for all $\psi \in \Psi_1$

$$\hat{s}(\psi) = \vee \{s(\chi) : \chi \in Y(s), \chi \ll \psi\},$$

where $\hat{s}(\psi) = 1$, if there is no $\chi \in Y(s)$ such that $\chi \ll \psi$. Then \hat{s} maps Ψ_1 into B_2 , and the range of \hat{s} is finite. We claim that $B = \{\hat{s} : s \in S\}$ is a basis in $([\Psi_1 \rightarrow \Psi_2]_c, E_1 \times E_2; \cdot, \circ)$. We show first that any map \hat{s} is continuous.

Lemma 8.2 *For every $s \in S$, the map \hat{s} is continuous.*

Proof. Obviously, \hat{s} is order-preserving. Consider a directed set X in Ψ_1 and an element $\psi \in X$. Then $\hat{s}(\psi) \leq \hat{s}(\bigvee X)$, hence $\bigvee_{\psi \in X} \hat{s}(\psi) \leq \hat{s}(\bigvee X)$. Conversely, consider an element $\psi \in Y(s)$ such that $\psi \ll \bigvee X$. Then there exists a $\phi \in X$ such that $\psi \leq \phi$, and $s(\psi) \leq \hat{s}(\psi) \leq \hat{s}(\phi)$. Thus, we see that

$$\hat{s}(\bigvee X) = \bigvee \{s(\psi) : \psi \in Y(s), \psi \ll \bigvee X\} \leq \bigvee_{\phi \in X} \hat{s}(\phi).$$

So we conclude that $\hat{s}(\bigvee X) = \bigvee_{\phi \in X} \hat{s}(\phi)$, which shows that \hat{s} is continuous. \square

Next, we verify that B is closed under combination. In fact, we have $\hat{s}_1 \cdot \hat{s}_2 = \hat{s}$, where

$$s(\psi) = \begin{cases} s_1(\psi) & \text{if } \psi \in Y(s_1) - Y(s_2), \\ s_2(\psi) & \text{if } \psi \in Y(s_2) - Y(s_1), \\ s_1(\psi) \vee s_2(\psi) & \text{if } \psi \in Y(s_1) \cap Y(s_2). \end{cases}$$

Finally we show that the maps $\hat{s} \ll f$ approximate the continuous map f . For this purpose we need the following lemma:

Lemma 8.3 *If $\hat{s}(\psi) \ll f(\psi)$ for all $\psi \in \Psi_1$, then $\hat{s} \ll f$.*

Proof. Consider a directed set G of continuous functions in $[\Psi_1 \rightarrow \Psi_2]_c$ such that $f \leq \bigvee G$. Then we have

$$f(\psi) \leq \bigvee_{g \in G} g(\psi)$$

for all $\psi \in \Psi_1$. Therefore, if $\hat{s}(\psi) \ll f(\psi)$, there is a map g_ψ such that $\hat{s}(\psi) \leq g_\psi(\psi)$. But \hat{s} takes on only finitely many different values χ_i . So, for each value $\chi_i = \hat{s}(\psi)$ we may select a map $g_i = g_\psi$ such that $\chi_i \leq g_i(\psi)$. Since G is directed, there is a map $g \in G$, such that $g_i \leq g$, hence $\chi_i \leq g$ for all i . But this means that $\hat{s} \leq g$ and this proves that $\hat{s} \ll f$. \square

Now, we are ready to show that $([\Psi_1 \rightarrow \Psi_2]_c; \leq)$ is a continuous lattice.

Theorem 8.3 *For every $f \in [\Psi_1 \rightarrow \Psi_2]_c$ we have*

$$f = \bigvee \{\hat{s} : \hat{s} \ll f\}. \quad (8.1)$$

Proof. We show $f(\psi) = \bigvee \{\hat{s}(\psi) : \hat{s} \ll f\}$ for all $\psi \in \Psi_1$. This proves then (8.1). So fix a $\psi \in \Psi_1$. By continuity in Ψ_2 we have

$$f(\psi) = \bigvee \{\chi \in B_2 : \chi \ll f(\psi)\} = \bigvee \{\hat{s}(\psi) : \hat{s}(\psi) \ll f(\psi)\},$$

since for all $\chi \in B_2$ there is a function \hat{s} such that $\hat{s}(\psi) = \chi$. In fact, by continuity in Ψ_1 we have $\psi = \bigvee \{\eta \in B_1 : \eta \ll \psi\}$. Select a $\eta \ll \psi$ and define for $Y = \{\eta\}$ the simple function $s(\eta) = \chi$. Then clearly $\hat{s}(\psi) = \chi$. Now, using Lemma 8.3 we conclude that

$$\begin{aligned} & \{\hat{s}(\psi) : \text{for } \hat{s} \text{ such that } \hat{s}(\psi) \ll f(\psi)\} \\ &= \{\hat{s}(\psi) : \hat{s}(\chi) \ll f(\chi) \text{ for all } \chi \in \Psi_1\} \subseteq \{\hat{s}(\psi) : \hat{s} \ll f\} \end{aligned}$$

So, we have

$$f(\psi) \leq \bigvee \{\hat{s}(\psi) : \hat{s} \ll f\} \leq f(\psi)$$

since $\hat{s} \ll f$ implies $\hat{s} \leq f$, hence $\hat{s}(\psi) \leq f(\psi)$. Therefore we obtain $f(\psi) = \bigvee \{\hat{s}(\psi) : \hat{s} \ll f\}$ for all $\psi \in \Psi_1$, hence (8.1). \square

Since in $[\Psi_1 \rightarrow \Psi_2]_c; \leq$ the order is point-wise, it follows that it is a complete lattice, hence a continuous lattice. This permits to conclude that continuous maps form a continuous valuation algebra.

Theorem 8.4 *If $(\Psi_1, E_1; \cdot, \circ)$ and $(\Psi_2, E_2; \cdot, \circ)$ are two continuous domain-free valuation algebras, then $([\Psi_1 \rightarrow \Psi_2]_c, E_1 \times E_2; \cdot, \circ)$ is a continuous domain-free valuation algebra.*

Proof. We know already that $([\Psi_1 \rightarrow \Psi_2]_c, E_1 \times E_2; \cdot, \circ)$ is an idempotent valuation algebra and $([\Psi_1 \rightarrow \Psi_2]_c; \leq)$ is a continuous lattice. Therefore, by Theorem 7.19 (which applies as well to idempotent valuation algebras as to generalised idempotent information algebras) we need only to prove that

$$(\epsilon_1, \epsilon_2)(\bigvee G) = \bigvee_{g \in G} (\epsilon_1, \epsilon_2)(g) \quad (8.2)$$

for any directed set G in $[\Psi_1 \rightarrow \Psi_2]_c$. This is straightforward using Theorem 7.19 in Ψ_2 ,

$$\begin{aligned} & (\epsilon_1, \epsilon_2)(\bigvee G)(\psi) \\ &= \epsilon_2((\bigvee G)(\epsilon_1(\psi))) = \epsilon_2(\bigvee_{g \in G} g(\epsilon_1(\psi))) \\ &= \bigvee_{g \in G} \epsilon_2(g(\epsilon_1(\psi))) = \bigvee_{g \in G} (\epsilon_1, \epsilon_2)(g)(\psi) \end{aligned}$$

for all $\psi \in \Psi_1$. This proves (8.2), hence the theorem. \square

The case, where $(\Psi_1, E_1; \cdot, \circ)$ and $(\Psi_2, E_2; \cdot, \circ)$ are algebraic has been discussed in (Kohlas, 2003a), although only in the multivariate setting. It has there that in this case $([\Psi_1 \rightarrow \Psi_2]_c, E_1 \times E_2; \cdot, \circ)$ is also an algebraic valuation algebra with \hat{s} as finite elements.

8.2 Cartesian Closed Categories

We consider the categories of idempotent, domain-free valuation algebras **IA**, and of algebraic and continuous valuation algebra **ALGIA** and **CONTIA** and we are going to show that these categories are all Cartesian closed. We assume throughout in the sequel, that the valuation axioms satisfy axioms D0, D1 and D3,D4,D5, whereas the Support Axiom D2 is not required.

1. The category **IA** has as objects idempotent, domain-free valuation algebras and as morphisms *information maps* $\Phi \rightarrow \Psi$.
2. The category of *continuous* valuation algebras **CONTIA** has as objects continuous valuation algebras and as morphisms *continuous maps* $\Phi \rightarrow \Psi$.
3. The category of *algebraic* valuation algebras **ALGIA** has as objects algebraic valuation algebras and as morphisms *continuous maps* $\Phi \rightarrow \Psi$.

The category **ALGIA** is a subcategory of **CONTIA**, which itself is a subcategory of **IA**. We are going to show that all these categories are *Cartesian closed*. To remind: A category **C** is Cartesian closed, if it satisfies the following three conditions:

1. The category **C** has a *terminal object*: There is an object $T \in \mathbf{C}$ such that there is exactly one morphism from any object to T .
2. The category **C** has *finite products*: For any pair of objects $A, B \in \mathbf{C}$, there is an object $A \times B$ and morphisms $p_A; A \times B \rightarrow A$ and $p_B : A \times B \rightarrow B$, such for any object C and for pair of morphisms $f_1 : C \rightarrow A$ and $f_2 : C \rightarrow B$ there is a morphism $f : C \rightarrow A \times B$ so that $p_A \circ f = f_1$ and $p_B \circ f = f_2$.
3. The category **C** has *exponentials*: For any pair of objects $B, C \in \mathbf{C}$, there is an object C^B and a morphism $eval : C^B \times B \rightarrow C$ such that for every morphism $f : A \times B \rightarrow C$ there is a unique morphism $\lambda f : A \rightarrow C^B$ so that $eval \circ (\lambda f, id_B) = f$.

We are going to show that these elements exist for our three categories **IA**, **CONTIA** and **ALGIA**. The terminal object in all three cases is simply

the valuation algebra $(\{1\}, \{id\}; \cdot, \circ)$. The finite product is the Cartesian product of valuation algebras.

Theorem 8.5 *The Cartesian product $(\Psi_1 \times \Psi_2, E_1 \times E_2; \cdot, \circ)$ of two idempotent (continuous, algebraic) valuation algebras $(\Psi_1, E_1; \cdot, \circ)$ and $(\Psi_2, E_2; \cdot, \circ)$ under component-wise join and meet respectively and also component-wise information extraction, is the categorical direct product of the two valuation algebras in **IA** (**CONTIA**, **ALGIA**, respectively).*

Proof. We show first that $(\Psi_1 \times \Psi_2, E_1 \times E_2; \cdot, \circ)$ is an idempotent valuation algebra. Combination in $\Psi_1 \times \Psi_2$ is defined component-wise and it is obvious that $(\Psi_1 \times \Psi_2; \cdot)$ is then an idempotent commutative semigroup with null element $(0, 0)$ and unit $(1, 1)$, hence a bounded join-semilattice under the induced information order. In $E_1 \times E_2$ meet is defined component-wise and in this way $(E_1 \times E_2, \leq)$ becomes a meet-semilattice. For any pair (ϵ_1, ϵ_2) in $E_1 \times E_2$, an operator

$$(\epsilon_1, \epsilon_2)(\psi_1, \psi_2) = (\epsilon_1(\psi_1), \epsilon_2(\psi_2))$$

is defined. It is obvious that axioms D2 to D5 of an idempotent valuation algebra carry over to the Cartesian product $(\Psi_1 \times \Psi_2, E_1 \times E_2; \cdot, \circ)$, which is therefore an idempotent information algebra.

We define the projections p_i by $p_i(\psi_1, \psi_2) = \psi_i$ for $i = 1, 2$. These projections are clearly information maps. Consider then an idempotent information algebra $(\Psi, E; \cdot, \circ)$ and two information maps $f_i : \Psi \rightarrow \Psi_i$, for $i = 1, 2$. Define $f : \Psi \rightarrow \Psi_1 \times \Psi_2$ by $f(\psi) = (f_1(\psi), f_2(\psi))$. Again, f is an information map. Then, $f_i = p_i \circ f$ for $i = 1, 2$. Thus, the product algebra $(\Psi_1 \times \Psi_2, E_1 \times E_2; \cdot, \circ)$ is the direct product of $(\Psi_1, E_1; \cdot, \circ)$ and $(\Psi_2, E_2; \cdot, \circ)$ in **IA**.

Next, we show that the Cartesian product of two continuous valuation algebras is continuous. Let then B_1 and B_2 be bases in Ψ_1 and Ψ_2 respectively. Obviously $B_1 \times B_2$ is closed under join and contains the bottom element $(1_1, 1_2)$, where 1_1 and 1_2 are the bottom elements of Ψ_1 and Ψ_2 respectively. We claim that $B_1 \times B_2$ is a basis of $\Psi_1 \times \Psi_2$. Let $X \subseteq B_1 \times B_2$ be a directed set and define $X_1 = \{\psi_1 \in B_1 : \exists \psi_2 \in B_2 \text{ so that } (\psi_1, \psi_2) \in X\}$. X_2 is defined similarly as the set of elements in B_2 obtained from X . Both X_1 and X_2 are clearly directed. Then $(\bigvee X_1, \bigvee X_2)$ is an upper bound of X , and it is obviously its supremum. So $\bigvee X = (\bigvee X_1, \bigvee X_2)$ exists in $\Psi_1 \times \Psi_2$. This is the convergence property.

We have $(\psi'_1, \psi'_2) \ll (\psi_1, \psi_2)$ if and only if, $\psi'_1 \ll \psi_1$ and $\psi'_2 \ll \psi_2$, the \ll -relation taken in $\Psi_1 \times \Psi_2$, Ψ_1 and Ψ_2 respectively. Consider $(\psi_1, \psi_2) \in \Psi_1 \times \Psi_2$. Then

$$\begin{aligned} & \bigvee \{(\psi'_1, \psi'_2) \in B_1 \times B_2 : (\psi'_1, \psi'_2) \ll (\psi_1, \psi_2)\} \\ &= (\bigvee \{\psi'_1 \in B_1 : \psi'_1 \ll \psi_1\}, \bigvee \{\psi'_2 \in B_2 : \psi'_2 \ll \psi_2\}) \\ &= (\psi_1, \psi_2). \end{aligned}$$

This shows that $\Psi_1 \times \Psi_2$ is a continuous lattice.

If $(\epsilon_1, \epsilon_2) \in E_1 \times E_2$, then we obtain in the same way

$$\begin{aligned} & \bigvee \{(\psi'_1, \psi'_2) \in B_1 \times B_2 : (\psi'_1, \psi'_2) = (\epsilon_1, \epsilon_2)(\psi'_1, \psi'_2) \ll (\epsilon_1, \epsilon_2)(\psi_1, \psi_2)\} \\ &= (\bigvee \{\psi'_1 \in B_1 : \psi'_1 = \epsilon_1(\psi'_1) \ll \epsilon_1(\psi_1)\}, \\ & \bigvee \{\psi'_2 \in B_2 : \psi'_2 = \epsilon_2(\psi'_2) \ll \epsilon_2(\psi_2)\}) \\ &= (\epsilon_1(\psi_1), \epsilon_2(\psi_2)) = (\epsilon_1, \epsilon_2)(\psi_1, \psi_2). \end{aligned}$$

So strong density holds too. This proves that $(\Psi_1 \times \Psi_2, E_1 \times E_2; \cdot, \circ)$ is a continuous information algebra.

The projections p_1 and p_2 are evidently continuous maps. Let then finally $(\Psi, E; \cdot, \circ)$ be a continuous information algebra and f_1 and f_2 be continuous maps $f_1 : \Psi \rightarrow \Psi_1$ and $f_2 : \Psi \rightarrow \Psi_2$. Then we define $f = (f_1, f_2)$ as a map from Ψ to $\Psi_1 \times \Psi_2$. It is continuous, since its components f_1 and f_2 are so. Then clearly $p_1 \circ f = f_1$ and $p_2 \circ f = f_2$. It follows that $(\Psi_1 \times \Psi_2, E_1 \times E_2; \cdot, \circ)$ is the direct product in **CONTIA**.

If $(\Psi_1, E_1; \cdot, \circ)$ and $(\Psi_2, E_2; \cdot, \circ)$ are algebraic valuation algebras, then $(\Psi_1 \times \Psi_2, E_1 \times E_2)$ is algebraic, and its finite elements are given by the Cartesian product of the finite elements of each factor since $(\psi'_1, \psi'_2) \ll (\psi_1, \psi_2)$ exactly if $\psi'_1 \ll \psi_1$ and $\psi'_2 \ll \psi_2$. So, $(\Psi_1 \times \Psi_2, E_1 \times E_2; \cdot, \circ)$ is the direct product in **ALGIA**. This completes the proof. \square

Next we show that the valuation algebras of monotone or continuous maps are the exponentials of the respective category of idempotent, continuous or algebraic valuation algebras.

Theorem 8.6 *If $(\Psi_1, E_1; \cdot, \circ)$ and $(\Psi_2, E_2; \cdot, \circ)$ are two objects of the category **IA**, then the idempotent valuation algebra $([\Psi_1 \rightarrow \Psi_2], E_1 \times E_2; \cdot, \circ)$ is an exponential of **IA**. If $(\Psi_1, E_1; \cdot, \circ)$ and $(\Psi_2, E_2; \cdot, \circ)$ are two objects of the categories **CONTIA** or **ALGIA**, then $([\Psi_1 \rightarrow \Psi_2]_c, E_1 \times E_2; \cdot, \circ)$ is an exponential of the respective categories.*

Proof. We only treat the case of continuous valuation algebras, the other cases follow in the same way. We know from Theorem 8.4 that $([\Psi_1 \rightarrow \Psi_2], E_1 \times E_2; \cdot, \circ)$ is a continuous information algebra. We define the morphism $eval : [\Psi_1 \rightarrow \Psi_2]_c \times \Psi_1 \rightarrow \Psi_2$ for $f \in [\Psi_1 \rightarrow \Psi_2]_c$ and $\psi \in \Psi_1$ by

$$eval(f, \psi) = f(\psi).$$

The map $eval$ is continuous.

Consider another continuous valuation algebra $(\Psi, E; \cdot, \circ)$ and let $f : \Psi \times \Psi_1 \rightarrow \Psi_2$ be a continuous map. Then we define a map $\lambda f : \Psi \rightarrow [\Psi_1 \rightarrow \Psi_2]_c$ for $\chi \in \Psi$ and $\psi \in \Psi_1$ by

$$\lambda f(\chi)(\psi) = f(\chi, \psi).$$

The map λf is continuous if f is so. In fact, let X be a directed set in Λ . Then we have for $\psi \in \Psi_1$,

$$\lambda f(\bigvee X)(\psi) = f(\bigvee X, \psi) = f(\bigvee_{\chi \in X} (\chi, \psi)) = \bigvee_{\chi \in X} f(\chi, \psi) = \bigvee_{\chi \in X} \lambda f(\chi)(\psi).$$

Thus we see that $\lambda f(\bigvee X) = \bigvee_{\chi \in X} \lambda f(\chi)$.

Now finally for $(\chi, \psi) \in \Psi \times \Psi_1$, we obtain that $eval \circ (\lambda f, id_{\Psi_1})(\chi, \psi) = eval(\lambda f(\chi), \psi) = \lambda f(\chi)(\psi) = f(\chi, \psi)$. So indeed $eval \circ (\lambda f, id_{\Psi_1}) = f$.

The cases of idempotent and of algebraic valuation algebras are treated in exactly the same way. \square

This shows that the categories **IA**, **ALGIA** and **CONTIA** are all Cartesian closed.

Chapter 9

Random Maps

9.1 Simple Random Variables

In practice it can not be excluded that *contradictory information* is asserted. Then at least one of these assertions must be wrong. This immediately leads to the idea that information may be uncertain, at least in the sense that its assertion may be wrong. For instance, if the source of an information is a witness, an expert or a sensor, there is always the possibility that the witness lies, the expert errs or that the sensor is faulty. More generally, the truth of a piece of information may depend on certain assumptions whose validity is uncertain. Turned the other way round: Assuming certain assumptions out of a set of possible assumptions, certain pieces of information may be asserted. The uncertainty of the information stems from the uncertainty about which assumption is valid. Also different assumptions may have different likelihood or probabilities to be valid. Viewed from this angle, *uncertain information* is represented by a map from a probability space into an (idempotent) generalised information algebra.

Given such a map, for any piece of information in the information algebra, or more generally each consistent system of information in its ideal completion, the assumptions *supporting* the information considered can be determined: These are all the assumptions whose validity entails the information. The probability of the assumptions supporting a piece of information measures the degree of support of it. Here enters the question of the *measurability* of the support. To overcome the restrictions imposed by measurability considerations, allocations of probability in the probability algebra associated with the probability space of assumptions can be considered (Kappos, 1969; Shafer, 1973).

Maps representing uncertain information inherit the structure of an information algebra from their range. Uncertain information thus still is *information*. In many cases, *finite* uncertain information is in a natural way to be defined, which turns these algebras of uncertain information into compact

or algebraic information algebras.

The present concept of uncertain information has its roots in the *theory of hints* (Kohlas & Monney, 1995) which in turn is based on Dempster's multivalued mappings (Dempster, 1967a). However, whereas Dempster derives probability bounds from these multivalued mappings, the semantics of the theory of hints is in the spirit of assumption-based reasoning as sketched above. Seen from the point of view of information algebra, hints are mappings into a subset-algebra. The theory can also be given a logical flavour. It may for instance be combined with propositional logic (Haenni *et al.*, 2000; Kohlas, 2003a). Since this approach combines logic for deduction of arguments with probability to evaluate likelihood of arguments, we speak also of *probabilistic argumentation systems*. A more abstract presentation of this point of view is given in (Kohlas, 2003b).

Dempster's approach to multivalued mappings was given by Shafer a more epistemological flavor (Shafer, 1976). The primary object in this view is the *belief function* which corresponds formally to our degree of support and leads to an allocation of probability as hinted above (Shafer, 1973). Therefore, in the spirit of Shafer, we study allocations of belief and show that they too lead to information algebras (Section 10). In particular, we study how these allocations of probabilities relate to the mappings representing uncertain information.

We start with simple random variables. Consider a generalised, idempotent, domain-free information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. Similar as in the previous section, we require throughout this section only axioms A0, A1 and A3, A4, A5, A6 for a generalised information algebra and drop the Support Axiom AY2. Let Ω be a set whose elements represent different possible assumptions. In applications, Ω often will be a finite set. But we drop this requirement for the sake of generality. In order to introduce probability, we assume (Ω, \mathcal{A}, P) to be a *probability space* with \mathcal{A} a σ -algebra of subsets of Ω and P a probability measure on \mathcal{A} . Uncertain information will be represented by a map Δ from Ω to Ψ . The idea is that $\Delta(\omega) \in \Psi$ represents the correct information, provided assumption $\omega \in \Omega$ is valid. In order to simplify, and for considerations of measurability, which will be dropped later, we restrict however in a first step the maps to be considered. Let $\mathcal{B} = \{B_1, \dots, B_n\}$ be any finite partition of Ω , whose blocks B_i belong all to \mathcal{A} . A mapping $\Delta : \Omega \rightarrow \Psi$, such that $\Delta(\omega)$ is constant for all ω of a block B_i ,

$$\Delta(\omega) = \psi_i, \text{ for all } \omega \in B_i,$$

is called a *simple random variable* in Ψ .

Denote the family of all simple random variables by \mathcal{R}_s . These maps inherit the operations of the information algebra:

1. *Combination:* Let Δ_1 and Δ_2 be simple random variables in $(\Psi, D; \leq$

, \perp , \cdot , ϵ). Then $\Delta_1 \cdot \Delta_2$ is defined pointwise by

$$(\Delta_1 \cdot \Delta_2)(\omega) = \Delta_1(\omega) \cdot \Delta_2(\omega),$$

where on the right combination is in Ψ .

2. *Extraction*: Let Δ be a simple random variable in $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. Then define $\epsilon_x(\Delta)$ for $x \in D$ by

$$(\epsilon_x(\Delta))(\omega) = \epsilon_x(\Delta(\omega)),$$

where on the right extraction takes place in Ψ .

We have to verify that the maps so defined are still simple random variables. Let \mathcal{B}_1 and \mathcal{B}_2 be the finite partitions of Ω associated with Δ_1 and Δ_2 respectively. Then $\mathcal{B} = \mathcal{B}_1 \wedge \mathcal{B}_2$ is defined as the partition of Ω whose blocks are the pairwise intersections of blocks from \mathcal{B}_1 and \mathcal{B}_2 . Clearly, the map $\Delta_1 \cdot \Delta_2$ is constant on each block of \mathcal{B} , hence a simple random variable. If further Δ is defined relative to a partition \mathcal{B} of Ω , then $\epsilon_x(\Delta)$ is also constant on the blocks of \mathcal{B} , hence also a simple random variable. Obviously, $(\mathcal{R}_s, D; \leq, \perp, \cdot, \epsilon)$ becomes a generalised idempotent, domain-free information algebra with these operations. The null element is the simple random variable N defined by $N(\omega) = 0$, the unit element the simple random variable U defined by $U(\omega) = 1$ for all $\omega \in \Omega$. Furthermore, for every $\psi \in \Psi$ the map $D_\psi(\omega) = \psi$, for all $\omega \in \Omega$, is a simple random variable. By the mapping $\psi \mapsto D_\psi$ the information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is embedded in the information algebra $(\mathcal{R}_s, D; \leq, \perp, \cdot, \epsilon)$.

Note that the partial order in \mathcal{R}_s is also defined point-wise such that $\Delta_1 \leq \Delta_2$ in \mathcal{R}_s if and only if, $\Delta_1(\omega) \leq \Delta_2(\omega)$ for all $\omega \in \Omega$.

There are two important special classes of simple random variables: If for a random variable Δ defined relative to a partition $\mathcal{B} = \{B_1, \dots, B_n\}$ it holds that $\psi_i \neq \psi_j$ for $i \neq j$, the variable is called *canonical*. It is a simple matter to transform any random variable Δ into an associated canonical one: Take the union of all blocks $B_i \in \mathcal{B}$ with identical values ϕ_i . This yields a new partition \mathcal{B}' of Ω . Define $\Delta'(\omega) = \Delta(\omega)$. Then Δ' is the *canonical version* of Δ and we write $\Delta' = \Delta^\rightarrow$. We may consider the set of *canonical random variables*, $\mathcal{R}_{s,c}$, and define between elements of this set combination and extraction as follows:

$$\begin{aligned} \Delta_1 \cdot_c \Delta_2 &= (\Delta_1 \cdot \Delta_2)^\rightarrow, \\ \epsilon_{x,c}(\Delta) &= (\epsilon_x(\Delta))^\rightarrow. \end{aligned}$$

Then $(\mathcal{R}_{s,c}, D; \leq, \perp, \cdot_c, \epsilon_c)$ is still an information algebra under these modified operations. We remark also that $(\Delta_1 \cdot \Delta_2)^\rightarrow = (\Delta_1^\rightarrow \cdot \Delta_2^\rightarrow)^\rightarrow$ and $(\epsilon_x(\Delta))^\rightarrow = (\epsilon_x(\Delta^\rightarrow))^\rightarrow$. In fact, $(\mathcal{R}_{s,c}, D; \leq, \perp, \cdot_c, \epsilon_c)$ is the quotient algebra of $(\mathcal{R}_s, D; \leq, \perp, \cdot, \epsilon)$ relative to the congruence $\Delta_1 \equiv \Delta_2$, if $\Delta_1^\rightarrow = \Delta_2^\rightarrow$.

Secondly, if $\Delta(\omega) = 0$ with probability zero, then Δ is called *normalised*. We can associate a normalised simple random variables Δ^\downarrow with any simple random variable Δ provided $\Delta(\omega) \neq 0$ occurs with a positive probability. In fact, let $\Omega^\downarrow = \{\omega \in \Omega : \Delta(\omega) \neq 0\}$. This is a measurable set with probability $P(\Omega^\downarrow) = 1 - P\{\omega \in \Omega : \Delta(\omega) = 0\} > 0$. We consider then the new probability space $(\Omega, \mathcal{A}, P')$, where P' is the conditional probability measure on \mathcal{A} defined by

$$P'(A) = \frac{P(A \cap \Omega^\downarrow)}{P(\Omega^\downarrow)}, \quad (9.1)$$

if $A \cap \Omega^\downarrow \neq \emptyset$ and $P'(A) = 0$, otherwise. On this new probability space define $\Delta^\downarrow(\omega) = \Delta(\omega)$. Clearly, it holds that $(\Delta^\rightarrow)^\downarrow = (\Delta^\downarrow)^\rightarrow$.

The idea behind normalisation becomes clear, when we consider *combination* of random variables: Each of two (normalised) random variables Δ_0 and Δ_2 represents some (uncertain) information with the following interpretation: One of the $\omega \in \Omega$ must be the correct, but unknown assumption. However, if ω happens to be the correct assumption, then under the first random variable information $\Delta_1(\omega)$ can be asserted, and under the second variable information $\Delta_2(\omega)$. Thus, together, still under the assumption ω , information $\Delta_1(\omega) \cdot \Delta_2(\omega)$ can be asserted. However, it is possible that $\Delta_1(\omega) \cdot \Delta_2(\omega) = 0$, even if both Δ_1 and Δ_2 are *normalised*. But the element 0 represents a *contradiction*. Thus in view of the information given by the variables Δ_0 and Δ_2 , the assumption ω can not hold, since it leads to a contradiction; it can (and must) be excluded. This amounts to normalise the random variable $\Delta_1 \cdot \Delta_2$, by excluding all $\omega \in \Omega$ for which the combination results in a contradiction, and then to condition (i.e. normalise) the probability on non-contradictory assumptions. We refer to (Kohlas & Monney, 1995; Haenni *et al.*, 2000) for a discussion and further justification of these issues.

Two partitions \mathcal{B}_1 and \mathcal{B}_2 of Ω are called *independent*, if $B_{1,i} \cap B_{2,j} \neq \emptyset$ for all blocks $B_{1,i} \in \mathcal{B}_1$ and $B_{2,j} \in \mathcal{B}_2$. If furthermore $P(B_{1,i} \cap B_{2,j}) = P(B_{1,i}) \cdot P(B_{2,j})$ for all these pairs of blocks, then the two partitions \mathcal{B}_1 and \mathcal{B}_2 are called *stochastically independent*. In addition, if Δ_1 and Δ_2 are two simple random variables defined on these two partitions respectively, then these random variables are called *stochastically independent* too. Note that if Δ_1 and Δ_2 are stochastically independent, then their canonical versions Δ_1^\rightarrow and Δ_2^\rightarrow are stochastically independent too.

We now turn to the study of the *probability distribution* of simple random variables. The starting point is the following question: Given a simple random variable Δ in an idempotent, generalised information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$, and an element $\psi \in \Psi$, under what assumptions can the information ψ be asserted to hold? And how likely is it, that these assumptions are valid?

If $\omega \in \Omega$ is an assumption such that $\Delta(\omega) \geq \psi$, then $\Delta(\omega)$ implies ψ . In this case we may say that ω is an assumption *supporting* ψ , in view of the

information conveyed by Δ . Therefore we define for every $\psi \in \Psi$ the set

$$qs_{\Delta}(\psi) = \{\omega \in \Omega : \psi \leq \Delta(\omega)\}$$

of assumptions supporting ψ . However, if $\Delta(\omega) = 0$, then ω is supporting every $\psi \in \Psi$, since $\psi \leq 0$. The null element 0 represents the *contradiction*, which implies everything. In a consistent theory, contradictions must be excluded. Thus, we conclude that assumptions such that $\Delta(\omega) = 0$ are not really possible assumptions and must be excluded. Let

$$qs_{\Delta}(0) = \{\omega \in \Omega : \Delta(\omega) = 0\}.$$

We assume that $qs_{\Delta}(0)$ is not equal to Ω ; otherwise Δ is representing fully contradictory “information”. In other words, we assume that proper information is never fully contradictory. If we eliminate the contradictory assumptions from $qs(\phi)$, we obtain the *support set*

$$s_{\Delta}(\psi) = \{\omega \in \Omega : \psi \leq \Delta(\omega) \neq 0\} = qs_{\Delta}(\psi) - qs_{\Delta}(0).$$

of ψ , which is the set of assumptions properly supporting ψ and the mapping $s_{\Delta} : \Psi \rightarrow \mathbb{P}(\Omega)$ is called the *allocation of support* induced by Δ . The set $qs(\psi)$ is called the *quasi-support set* to underline that it contains contradictory assumptions. This set has little interest from a semantic point of view, but it is useful for technical and especially for computational purposes. These concepts capture the essence of probabilistic assumption-based reasoning in information algebras as discussed in more detail in (Kohlas & Monney, 1995; Haenni *et al.*, 2000; Kohlas, 2003a) in a less general setting.

Here are the basic properties of allocations of support:

Theorem 9.1 *If Δ is a simple random variable on an idempotent generalised information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$, then the following holds for the associated allocation of support qs_{Δ}, s_{Δ} :*

1. $qs_{\Delta}(1) = \Omega, s(0) = \emptyset$.
2. If Δ is normalised, then $qs_{\Delta} = s_{\Delta}$ and $qs_{\Delta}(0) = \emptyset$.
3. For any pair $\phi, \psi \in \Phi$,

$$\begin{aligned} qs_{\Delta}(\phi \cdot \psi) &= qs_{\Delta}(\phi) \cap qs_{\Delta}(\psi), \\ s_{\Delta}(\phi \cdot \psi) &= s_{\Delta}(\phi) \cap s_{\Delta}(\psi). \end{aligned}$$

Proof. (1) and (2) follow immediately from the definition of the allocation of support. (3) follows since $\phi \cdot \psi \leq \Delta(\omega)$ if and only if $\phi \leq \Delta(\omega)$ and $\psi \leq \Delta(\omega)$. \square

Knowing assumptions supporting a hypothesis ψ is already interesting and important. It is the part logic can provide. On top of this, it is important

to know how *likely* it is that a supporting assumption is valid. This is the part added by probability. If we know or may assume that the information is consistent, then we should condition the original probability measure P in Ω on the event $qs_{\Delta}^c(0)$. This leads then to the probability space $(qs_{\Delta}^c(0), \mathcal{A} \cap qs_{\Delta}^c(0), P')$, where $P'(A) = P(A)/P(qs_{\Delta}^c(0))$. The likelihood of supporting assumptions for $\psi \in \psi$ can then be measured by

$$sp_{\Delta}(\psi) = P'(s_{\Delta}(\psi)).$$

The value $sp_{\Delta}(\psi)$ is called the *degree of support* of ψ associated with the random variable Δ . The function $sp : \psi \rightarrow [0, 1]$ is called the *support function* of Δ . It corresponds to the concept of a *distribution function* of ordinary random variables.

It is for technical reasons convenient to define the *degree of quasi-support*

$$qsp_{\Delta}(\psi) = P(qs_{\Delta}(\psi)).$$

Then, the degree of support can also be expressed in terms of degrees of quasi-support

$$sp_{\Delta}(\psi) = \frac{qsp_{\Delta}(\psi) - qsp_{\Delta}(0)}{1 - qsp_{\Delta}(0)}.$$

This is the form which is usually used in applications (Haenni *et al.*, 2000).

In another consideration, we can also ask for assumptions $\omega \in \Omega$, under which Δ shows ψ to be possible, that is, not excluded, although not necessarily supported. If $\Delta(\omega)$ is such that combined with ψ it leads to a contradiction, i.e. if $\Delta(\omega) \cdot \psi = 0$, then under ω the information ψ is excluded by a consistency consideration as above. So we define the set

$$p_{\Delta}(\psi) = \{\omega \in \Omega : \Delta(\omega) \cdot \psi \neq 0\}.$$

This is the set of assumptions under which ψ is not excluded, hence can be considered as possible. Therefore we call it the *possibility set* of ψ . Note that $p_{\Delta}(\psi) \subseteq qs_{\Delta}^c(0)$. We can then define the *degree of possibility*, also sometimes called *degree of plausibility* (e.g. in (Shafer, 1976)), by

$$pl_{\Delta}(\psi) = P'(p_{\Delta}(\psi)).$$

If $\omega \in qs_{\Delta}^c(0) - p_{\Delta}(\psi)$, then, under this assumption, ψ is impossible, that is contradictory with $\Delta(\omega)$. So the set $qs_{\Delta}^c(0) - p_{\Delta}(\psi)$ contains arguments *against* ψ and

$$do_{\Delta}(\psi) = P'(qs_{\Delta}^c(0) - p_{\Delta}(\psi)) = 1 - pl_{\Delta}(\psi).$$

can be called the *degree of doubt* in ψ . Note that $s_{\Delta}(\psi) \subseteq p_{\Delta}(\psi)$ since $\psi \leq \Delta(\omega) \neq 0$ implies $\psi \cdot \Delta(\omega) = \Delta(\omega) \neq 0$. Hence, we see that for

all $\psi \in \Psi$ we have that $sp_\Delta(\psi) \leq pl_\Delta(\psi)$. These considerations put simple random variables in the realm of the so-called Dempster-Shafer theory (Dempster, 1967b; Shafer, 1976).

To underline this further, consider for a simple random variable Δ with possible values ψ_1, \dots, ψ_n the probabilities

$$m(\psi_i) = \sum_{j: \psi_j = \psi_i} P(B_j).$$

Note that $m(\psi_i) = P(B_i)$, if the random variable Δ is canonical. Remark also that

$$\sum_{i=1}^n m(\psi_i) = 1.$$

Such a finite collection of probabilities $m(\psi_i)$ summing up to one for $i = 1, \dots, n$ is called a *basic probability assignment (bpa)* in Ψ . Since $qs_\Delta(\psi) = \cup_{\psi \leq \psi_i} B_i$ and $pl_\Delta(\psi) = \cup_{\psi \cdot \psi_i \neq 0} B_i$, we see that

$$qs_\Delta(\psi) = \sum_{\psi \leq \psi_i} m(\psi_i), \quad pl_\Delta(\psi) = \sum_{\psi \cdot \psi_i \neq 0} m(\psi_i).$$

So, the bpa of a simple random variable determines its degrees of support and plausibilities. In (Shafer, 1976), support functions are called *belief functions*. Furthermore, if Δ_1 and Δ_2 are two *stochastically independent* simple random variables with possible values $\psi_{1,1}, \dots, \psi_{1,n}$ and $\psi_{2,1}, \dots, \psi_{2,m}$, then the possible values of the combined random variable $\Delta = \Delta_1 \cdot \Delta_2$ are ψ_k , where each ψ_k is equal to a combination $\psi_{1,i} \cdot \psi_{2,j}$. Therefore, the bpa of the combined variable Δ is

$$m(\psi_k) = \sum_{\psi_{1,i} \cdot \psi_{2,j} = \psi_k} m_1(\psi_{1,i}) \cdot m_2(\psi_{2,j}).$$

If only *normalised* random variables are considered, then the combined variable Δ is to be normalised. Then, if

$$m(0) = \sum_{\psi_{1,i} \cdot \psi_{2,j} = 0} m_1(\psi_{1,i}) \cdot m_2(\psi_{2,j}) < 1,$$

we obtain the normalised bpa of Δ^\downarrow as

$$m^\downarrow(\psi_k) = \frac{\sum_{\psi_{1,i} \cdot \psi_{2,j} = \psi_k} m_1(\psi_{1,i}) \cdot m_2(\psi_{2,j})}{1 - m(0)} \quad (9.2)$$

So, the bpa are also sufficient to compute the bpa of the combination of stochastically independent pieces of uncertain information. This has been proposed in a setting of set algebras in (Dempster, 1967a) and the formula

(9.2) is therefore also called *Dempster's rule*. (Shafer, 1976) took up Dempster's theory and proposed "A Mathematical Theory of Evidence" where bpa and Dempster's rule play an import role. In both theories the concept of a bpa is central. Although Dempster's and Shafer's interpretation of the theory are not quite the same, one speaks often of the *Dempster-Shafer Theory*. At least the underlying mathematics in both views are identical. We shall argue in this chapter that our present theory is a natural generalisation of Dempster-Shafer theory which was confined essentially to finite subset algebras and simple random variables (in our terminology). However bpa can no more play the same basic role relative to generalised information algebras as in classical Dempster-Shafer theory, since bpa works only of simple random variables, but not for more general uncertain information. Also, the full flavour of the duality relation between support and plausibility is deployed only in the case of Boolean information algebras (Section 11.5).

9.2 Random Mappings

When we want to go beyond simple random mappings, there are several ways to do this. The most radical one is to consider any mapping $\Gamma : \Omega \rightarrow \Psi$ from a probability space (Ω, \mathcal{A}, P) into an idempotent, generalised information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ or may be even its ideal completion. Let's call such maps *random mappings*. As before, in the case of simple random variables, we may define the operations of combination and extraction between random mappings point-wise in $(\Psi, D; \leq, \perp, \cdot, \epsilon)$:

1. *Combination*: Let Γ_1 and Γ_2 be two random mappings into $(\Psi, D; \leq, \perp, \cdot, \epsilon)$, then $\Gamma_1 \cdot \Gamma_2$ is the random mapping into $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ defined by

$$(\Gamma_1 \cdot \Gamma_2)(\omega) = \Gamma_1(\omega) \cdot \Gamma_2(\omega). \quad (9.3)$$

2. *Extraction*: Let Γ be a random mapping into $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ and $x \in D$, then $\epsilon_x(\Gamma)$ is the random mapping into $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ defined by

$$\epsilon_x(\Gamma)(\omega) = \epsilon_x(\Gamma(\omega)). \quad (9.4)$$

For a fixed probability space (Ω, \mathcal{A}, P) , let \mathcal{R}_Ψ denote the set of all random mappings in $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. With the two operations defined above, $(\mathcal{R}_\Psi, D; \leq, \perp, \cdot, \epsilon)$ becomes an *idempotent generalised information algebra* (excluding the Support Axiom). The mapping $U(\omega) = 1$ for all $\omega \in \Omega$ is the neutral element of combination; the map $N(\omega) = 0$ the top element. It is obvious that $\Gamma' \leq \Gamma$ if and only if $\Gamma'(\omega) \leq \Gamma(\omega)$ for all $\omega \in \Omega$.

Consider the ideal completion $(I_{\mathcal{R}_\Psi}, D; \leq, \perp, \cdot, \epsilon)$ of the information algebra $(\mathcal{R}_\Psi, D; \leq, \perp, \cdot, \epsilon)$ of random mappings. Any element $\Gamma \in I_{\mathcal{R}_\Psi}$ may be represented as

$$\Gamma = \bigvee \{\Delta : \Delta \in \mathcal{R}_\Psi, \Delta \leq \Gamma\},$$

Obviously, we also have in the ideal completion I_Ψ of Ψ ,

$$\Gamma(\omega) = \bigvee \{\Delta(\omega) : \Delta \in \mathcal{R}_\Psi, \Delta \leq \Gamma\} = \bigvee \{\psi : \psi \in \Psi, \psi \leq \Gamma(\omega)\}$$

for all $\omega \in \Omega$. Therefore, $\Gamma \in I_{\mathcal{R}_\Psi}$ is also a random mapping into the ideal completion I_Ψ of the information algebra Ψ . And any random mapping into I_Ψ belongs to $I_{\mathcal{R}_\Psi}$. This shows that $(\mathcal{R}_{I_\Psi}, D; \leq, \perp, \cdot, \epsilon)$ is identical to the ideal completion $(I_{\mathcal{R}_\Psi}, D; \leq, \perp, \cdot, \epsilon)$ of the algebra $(\mathcal{R}_\Psi, D; \leq, \perp, \cdot, \epsilon)$.

As in the case of simple random variables we may define the allocation of support s_Γ of a random mapping by

$$s_\Gamma(\psi) = \{\omega \in \Omega : \psi \leq \Gamma(\omega)\}. \quad (9.5)$$

We do not any more distinguish here between the semantic categories of support and quasi-support as above for simple random variables and speak simply of support, even though (9.5) is strictly speaking a quasi-support.

This support, as defined in (9.5), has the same properties as the support of simple random variables, in particular, as in Theorem 9.1, $s_\Gamma(1) = \Omega$ and $s_\Gamma(\phi \cdot \psi) = s_\Gamma(\phi) \cap s_\Gamma(\psi)$. Again, as before, with simple random variables, we may try to define the degree of support induced by a random mapping Γ of a piece of information ψ by

$$sp_\Gamma(\psi) = P(s_\Gamma(\psi)). \quad (9.6)$$

This probability is however only defined if $s_\Gamma(\psi) \in \mathcal{A}$. There is no guarantee that this holds in general. The only element which we know for sure to be measurable is $s_\Gamma(1) = \Omega$. A simple way out of this problem would be to restrict random mappings to mappings Γ for which $s_\Gamma(\psi) \in \mathcal{A}$ for all $\psi \in \Psi$ or even for all elements of the ideal completion I_Ψ . However, there is a priori no reason why we should restrict ourselves exactly to those mappings. Therefore we prefer other, more rational approaches to overcome the difficulty of an only partial definition of degrees of support. Here we propose a first solution. Later we present some alternatives.

(Shafer, 1979) advocates the use of *probability algebras* instead of probability spaces as a natural framework for studying belief functions. Since degrees of support are like belief functions, we adapt this idea here. First, we introduce the probability algebra associated with a probability space (Kappos, 1969). Let \mathcal{J} be the σ -ideal of P -null sets in the σ -algebra \mathcal{A} of the probability space. Two sets $A', A'' \in \mathcal{A}$ are equivalent modulo \mathcal{J} , if

$A' - A'' \in \mathcal{J}$ and $A'' - A' \in \mathcal{J}$. This means that the two sets have the same probability measure $P(A') = P(A'')$. This equivalence is a congruence in the Boolean algebra \mathcal{A} . Hence the quotient algebra $\mathcal{B} = \mathcal{A}/\mathcal{J}$ is a Boolean σ -algebra too. If $[A]$ denotes the equivalence class of A , then, for any *countable* family of sets $A_i, i \in I$,

$$\begin{aligned} [A]^c &= [A^c], \\ \bigvee_{i \in I} [A_i] &= \left[\bigcup_{i \in I} A_i \right], \\ \bigwedge_{i \in I} [A_i] &= \left[\bigcap_{i \in I} A_i \right]. \end{aligned} \quad (9.7)$$

So $[A]$ defines a Boolean homomorphism from \mathcal{A} onto \mathcal{B} , called *projection*. We denote $[\Omega]$ by \top and $[\emptyset]$ by \perp . These are of course the top and bottom elements of \mathcal{B} . Now, as is well known, \mathcal{B} has some further important properties (see (Halmos, 1963)): It satisfies the *countable chain condition*, which means that any family of disjoint elements of \mathcal{B} is countable. Further, any Boolean algebra \mathcal{B} satisfying the countable chain condition is *complete*. That is, any subset $E \subseteq \mathcal{B}$ has a supremum $\bigvee E$ and an infimum $\bigwedge E$ in \mathcal{B} . Furthermore, the countable chain condition implies also that there is always a *countable* subset D of E with the same supremum and infimum, i.e. $\bigvee D = \bigvee E$ and $\bigwedge D = \bigwedge E$. We refer to (Halmos, 1963) for these results. Finally, by $\mu([A]) = P(A)$ a *normalised, positive measure* μ is defined on \mathcal{B} . Positive means here that $\mu(b) = 0$ implies $b = \perp$. A pair (\mathcal{B}, μ) of a Boolean σ -algebra \mathcal{B} , satisfying the countable chain condition, and a normalised, positive measure μ on it, is called a *probability algebra*.

We use now this construction of a probability algebra from a probability space to extend the definition of the degrees of support s_Γ beyond elements ψ for which $s_\Gamma(\psi)$ are measurable. Even if $s_\Gamma(\psi)$ is not measurable, any $A \in \mathcal{A}$ such that $A \subseteq s_\Gamma(\psi)$ represents an argument for ψ , that is a set of assumptions which supports ψ . To exploit this remark, define for every set $H \in \mathcal{P}(\Omega)$

$$\rho_0(H) = \bigvee \{[A] : A \subseteq H, A \in \mathcal{A}\}. \quad (9.8)$$

This mapping has interesting properties as the following theorem shows.

Theorem 9.2 *The application $\rho_0 : \mathcal{P}(\Omega) \rightarrow \mathcal{A}/\mathcal{J}$ as defined in (9.8) has the following properties:*

$$\begin{aligned} \rho_0(\Omega) &= \top, \\ \rho_0(\emptyset) &= \perp, \\ \rho_0\left(\bigcap_{i \in I} H_i\right) &= \bigwedge_{i \in I} \rho_0(H_i). \end{aligned} \quad (9.9)$$

if $\{H_i, i \in I\}$ is a countable family of subsets of Ω .

Proof. Clearly, $\rho_0(\Omega) = [\Omega] = \top \in \mathcal{A}/\mathcal{J}$. Similarly, $\rho_0(\emptyset) = [\emptyset] = \perp \in \mathcal{A}/\mathcal{J}$.

In order to prove the remaining identity, let $H_i, i \in I$ be a countable family of subsets of Ω . For every index i , there is a countable family of sets $H'_j \in \mathcal{A}$ such that $H'_j \subseteq H_i$ and $\rho_0(H_i) = \bigvee [H'_j] = [\bigcup H'_j]$ since \mathcal{B} satisfies the countable chain condition. Take $A_i = \bigcup H'_j$. Then $A_i \subseteq H_i$, $A_i \in \mathcal{A}$ and $P(A_i) = \mu(\rho_0(H_i))$. Define $A = \bigcap_{i \in I} A_i \in \mathcal{A}$. It follows that $A \subseteq \bigcap_{i \in I} H_i$ and, because the projection is a σ -homomorphism, we obtain $[A] = \bigwedge_{i \in I} [A_i] = \bigwedge_{i \in I} \rho_0(H_i)$.

We are going to show now that $[A] = \rho_0(\bigcap_{i \in I} H_i)$ which proves then the theorem. For this, it is sufficient to show that $P(A) = \mu(\rho_0(\bigcap_{i \in I} H_i))$. This is so, since $P(A) = \mu([A])$ and $A \subseteq \bigcap H_i$, hence $[A] \leq \rho_0(\bigcap H_i)$. Therefore, if $\mu([A]) = \mu(\rho_0(\bigcap H_i))$ we must well have $[A] = \rho_0(\bigcap H_i)$, since μ is positive.

Now, clearly $P(A) \leq \mu(\rho_0(\bigcap H_i))$. As above, we conclude that there is an $A' \in \mathcal{A}$, $A' \subseteq \bigcap H_i$ such that $P(A') = \mu(\rho_0(\bigcap H_i))$. Further, $A' \cup (A - A') \subseteq \bigcap H_i$ implies that $P(A' \cup (A - A')) = P(A')$, hence $P(A - A') = 0$. Define $A'_i = A_i \cup (A - A') \subseteq H_i$. Then $A_i - A'_i = \emptyset$ and therefore,

$$\begin{aligned} \mu(\rho_0(H_i)) &= P(A_i) \leq P(A'_i) = P(A_i) + P(A'_i - A_i) \\ &\leq \mu(\rho_0(H_i)). \end{aligned} \quad (9.10)$$

This implies that $P(A'_i - A_i) = 0$, therefore we have $[A_i] = [A'_i]$. Further

$$\begin{aligned} \bigcap A'_i &= \bigcap (A_i \cup (A' - A)) = (A' - A) \cup \left(\bigcap A_i \right) \\ &= (A' - A) \cup A = A \cup A' = A' \cup (A - A'). \end{aligned}$$

But $\bigcap A'_i$ and $\bigcap A_i$ are equivalent, since $[\bigcap A'_i] = \bigwedge [A'_i] = \bigwedge [A_i] = [\bigcap A_i]$. This implies finally that $P(A) = P(\bigcap A_i) = P(\bigcap A'_i) = P(A') + P(A - A') = P(A') = \mu(\rho_0(\bigcap H_i))$. This is what was to be proved. \square

Take now $\mathcal{B} = \mathcal{A}/\mathcal{J}$ and consider the probability algebra (\mathcal{B}, μ) . Then we compose the allocation of support s from Ψ into the power set $\mathcal{P}(\Omega)$ with the mapping ρ_0 from $\mathcal{P}(\Omega)$ into \mathcal{B} to a mapping $\rho = \rho_0 \circ s : \Psi \rightarrow \mathcal{B}$. Now we see that

$$\begin{aligned} \rho(1) &= \rho_0(s(1)) = \rho_0(\Omega) = \top, \\ \rho(\phi \cdot \psi) &= \rho_0(s(\phi \cdot \psi)) = \rho_0(s(\phi) \cap s(\psi)) \\ &= \rho_0(s(\phi)) \wedge \rho_0(s(\psi)) = \rho(\phi) \wedge \rho(\psi). \end{aligned} \quad (9.11)$$

A mapping ρ satisfying these two properties is called an *allocation of probability (a.o.p)* on the information algebra Ψ . In fact, it allocates an element

of the probability algebra \mathcal{B} to any element of the algebra Ψ . In this way, a random mapping Γ leads always to an allocation of probability $\rho_\Gamma = \rho_0 \circ s_\Gamma$, once a probability measure on the assumptions is introduced.

In particular, we may now define the degree of support for any $\psi \in \Psi$ by

$$sp_\Gamma(\psi) = \mu(\rho_\Gamma(\psi)). \quad (9.12)$$

This extends the support function (9.6) to all elements ψ of Ψ .

In this way, the degree of support $sp_\Gamma(\psi)$ is, according to (9.8), equal to the probability of the supremum of all $[A]$, where A is measurable and supports ψ . This can also be expressed in another way. In order to see this, we note an important property of probability algebras: Clearly $\mu(\bigwedge b_i) \leq \inf_i \mu(b_i)$ and $\mu(\bigvee b_i) \geq \sup_i \mu(b_i)$ holds for any family of elements $\{b_i\}$. But there are important cases where equality hold (Halmos, 1963). A subset D of \mathcal{B} is called *downward (upward) directed*, if for every pair $b', b'' \in D$ there is an element $b \in D$ such that $b \leq b' \wedge b''$ ($b \geq b' \vee b''$).

Lemma 9.1 *If D is a downward (upward) directed subset of \mathcal{B} , then*

$$\mu(\bigwedge_{i \in D} b_i) = \inf_{i \in D} \mu(b_i), \quad \left(\mu(\bigvee_{i \in D} b_i) = \sup_{i \in D} \mu(b_i) \right) \quad (9.13)$$

Proof. There is a countable subfamily of elements $c_i \in D$, $i = 1, 2, \dots$, which have the same meet as D . Define $c'_1 = c_1$ and select elements c'_i in the downward directed set D such that $c'_2 \leq c'_1 \wedge c_2$, $c'_3 \leq c'_2 \wedge c_3, \dots$. Then $c'_1 \geq c'_2 \geq c'_3 \geq \dots$ and this sequence has still the same infimum. However, by the continuity of probability we have

$$\mu(\bigwedge b_i) = \mu(\bigwedge c'_i) = \lim_{i \rightarrow \infty} \mu(c'_i) \geq \inf_i \mu(b_i). \quad (9.14)$$

But this implies $\mu(\bigwedge b_i) = \inf_i \mu(b_i)$. The case of upwards directed sets is proved in the same way. \square

Note now that $\{[A] : A \subseteq H, A \in \mathcal{A}\}$ is an upward directed family in \mathcal{B} . Therefore, according to Lemma 9.1 we have

$$\begin{aligned} sp_\Gamma(\psi) &= \mu(\rho_\Gamma(\psi)) = \mu(\bigvee \{[A] : A \in \mathcal{A}, A \subseteq s_\Gamma(\psi)\}) \\ &= \sup \{ \mu([A]) : A \in \mathcal{A}, A \subseteq s_\Gamma(\psi) \} \\ &= \sup \{ P(A) : A \in \mathcal{A}, A \subseteq s_\Gamma(\psi) \} \\ &= P_*(s_\Gamma(\psi)), \end{aligned} \quad (9.15)$$

where P_* is the inner probability measure associated with P . This shows, that the degree of support of a piece of information ψ defined by (9.12) is the inner probability of the support $s_\Gamma(\psi)$. Note that definitions (9.12) and (9.6)

coincide, if $s_\Gamma(\psi) \in \mathcal{A}$. Support functions and inner probability measures are thus closely related. This result is very appealing: any measurable set A , which is contained in $s_\Gamma(\psi)$ supports ψ . So we expect $P(A) \leq sp_\Gamma(\psi)$. In the absence of further information, it is reasonable to take $sp_\Gamma(\psi)$ to be the least upper bound of the probabilities of A supporting ψ .

A similar consideration can be made with respect to the possibility sets associated with elements of Φ with respect to a random mapping Γ . As before we define the possibility set of ψ as

$$p_\Gamma(\psi) = \{\omega \in \Omega : \Gamma(\omega) \cdot \psi \neq 0\}.$$

This set contains all assumptions ω which do not lead to a contradiction with ψ under the mapping Γ . Thus, the probability of this set, if it is defined, measures the *degree of possibility* or the *degree of plausibility* of ψ ,

$$pl_\Gamma(\psi) = P(p_\Gamma(\psi)). \quad (9.16)$$

As in the case of the degree of support, there is no guarantee that $p_\Gamma(\psi)$ is \mathcal{A} -measurable. But we can solve this problem in a way similar to the case of the degree of support. A measurable set $A \subseteq p_\Gamma^c(\psi)$ can be seen as an argument *against* the hypothesis ψ , in particular, if Γ is normalised. But $A \subseteq p_\Gamma^c(\psi)$ is equivalent to $A^c \supseteq p_\Gamma(\psi)$. So a measurable set $A \supseteq p_\Gamma(\psi)$ can be considered as an argument that hypothesis ψ cannot be excluded. Therefore we define for every set $H \in \mathcal{P}$

$$\xi_0(H) = \bigwedge \{[A] : A \supseteq H, A \in \mathcal{A}\}. \quad (9.17)$$

Note that $A \supseteq H$ if and only if $A^c \subseteq H^c$. This implies that $\xi_0(H) = (\rho_0(H^c))^c$. From this in turn we conclude that the following corollary to Theorem 9.2 holds:

Corollary 9.1 *The application $\xi_0 : \mathcal{P}(\Omega) \rightarrow \mathcal{A}/\mathcal{J}$ as defined in (9.17) has the following properties:*

$$\begin{aligned} \xi_0(\Omega) &= \top, \\ \xi_0(\emptyset) &= \perp, \\ \xi_0\left(\bigcup_{i \in I} H_i\right) &= \bigvee_{i \in I} \xi_0(H_i). \end{aligned} \quad (9.18)$$

if $\{H_i, i \in I\}$ is a countable family of subsets of Ω .

As before we can now compose p_Γ with ξ_0 to obtain a mapping $\xi_\Gamma = \xi_0 \circ p_\Gamma : \Phi \rightarrow \mathcal{B} = \mathcal{A}/\mathcal{J}$. We may then define for any $\psi \in \Psi$ a degree of plausibility by

$$pl_\Gamma(\psi) = \mu(\xi_\Gamma(\psi)). \quad (9.19)$$

Using Lemma 9.1 we obtain also

$$pl_{\Gamma}(\psi) = \inf\{P(A) : A \in \mathcal{A}, A \supseteq p_{\Gamma}(\psi)\} = P^*(p_{\Gamma}(\psi)).$$

Here P^* is the outer probability measure of the set $p_{\Gamma}(\psi)$. Thus, if $p_{\Gamma}(\psi)$ is measurable, then $P^*(p_{\Gamma}(\psi)) = P(p_{\Gamma}(\psi))$, which shows that (9.19) defines in fact an extension of the plausibility defined by (9.16).

In the general case considered here, no properties comparable to those of support (for instance Theorem 9.1) exist for possibility sets and degrees of possibility. This notion gets its full power only in the case of Boolean information algebra, where it becomes a dual concept to support.

9.3 Random Variables

We propose now alternative approaches to define random variables in an information algebra. We start with the idempotent generalised information algebra $(\mathcal{R}_s, D; \leq, \perp, \cdot, \epsilon)$ of *simple random variables* with values in an idempotent generalised information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ and defined on a sample space (Ω, \mathcal{A}, P) , and consider the *ideal completion* of this algebra, rather than the algebra $(\mathcal{R}_{\Psi}, D; \leq, \perp, \cdot, \epsilon)$ considered before. Let \mathcal{R} denote the ideal completion of \mathcal{R}_s . Then $(\mathcal{R}, D; \leq, \perp, \cdot, \epsilon)$ is an algebraic information algebra with simple random variables \mathcal{R}_s as *finite* elements, see Section 7.2. We call the elements of \mathcal{R} *generalised random variables*. A generalised random variable is thus an ideal of simple random variables. As usual, we identify henceforth \mathcal{R}_s with its image in \mathcal{R} , that is, we identify the simple random variables $\Delta \in \mathcal{R}_s$ with their principal ideals $\downarrow \Delta$ in \mathcal{R} . We also write $\Delta \leq \Gamma$ for $\Delta \in \Gamma$, referring to the order in \mathcal{R} . So, for any $\Gamma \in \mathcal{R}$ we may within the algebra $(\mathcal{R}, D; \leq, \perp, \cdot, \epsilon)$ write $\Gamma = \bigvee \{\Delta \in \mathcal{R}_s : \Delta \leq \Gamma\}$. Using the associativity of join in the complete lattice \mathcal{R} , we obtain

$$\begin{aligned} \Gamma_1 \vee \Gamma_2 &= \Gamma_1 \cdot \Gamma_2 \\ &= \left(\bigvee \{\Delta_1 \in \mathcal{R}_s : \Delta_1 \leq \Gamma_1\} \right) \vee \left(\bigvee \{\Delta_2 \in \mathcal{R}_s : \Delta_2 \leq \Gamma_2\} \right) \\ &= \bigvee \{\Delta_1 \cdot \Delta_2 : \Delta_1, \Delta_2 \in \mathcal{R}_s, \Delta_1 \leq \Gamma_1, \Delta_2 \leq \Gamma_2\}. \end{aligned} \quad (9.20)$$

Note that this corresponds also to the combination of two ideals, see Section 7.1. In a similar way, by Theorem 7.2, we find that

$$\epsilon_x(\Gamma) = \epsilon_x\left(\bigvee \{\Delta \in \mathcal{R}_s : \Delta \leq \Gamma\}\right) = \bigvee \{\epsilon_x(\Delta) : \Delta \in \mathcal{R}_s : \Delta \leq \Gamma\}. \quad (9.21)$$

Again, this corresponds to the definition of extraction in the ideal completion, Section 7.1.

To any $\Gamma \in \mathcal{R}$ we may associate a random mapping $\Gamma : \Omega \rightarrow I_{\Psi}$ from the underlying sample space into the ideal completion of Ψ by defining

$$\Gamma(\omega) = \bigvee \{\Delta(\omega) : \Delta \in \mathcal{R}_s, \Delta \leq \Gamma\}. \quad (9.22)$$

This random mapping is defined by a sort of *point-wise limit* within I_Φ . We denote the random mapping Γ deliberately with the same symbol as the generalised random variable Γ . The reason is that the two concept can essentially be identified as the following lemmata show. In the following lemma, combination and extraction in \mathcal{R} are defined as in (9.20) and (9.21). Note that we denote combination (join) and information extraction for $x \in D$ with the same symbol in \mathcal{R} and in I_Ψ .

Lemma 9.2 1. If $\Gamma_1, \Gamma_2 \in \mathcal{R}$, then

$$(\Gamma_1 \cdot \Gamma_2)(\omega) = \Gamma_1(\omega) \cdot \Gamma_2(\omega) \text{ for all } \omega \in \Omega.$$

2. If $\Gamma \in \mathcal{R}$ and $\forall x \in D$, then

$$(\epsilon_x(\Gamma))(\omega) = \epsilon_x(\Gamma(\omega)) \text{ for all } \omega \in \Omega.$$

Proof. (1) By definition of the random mapping associated with $\Gamma_1 \cdot \Gamma_2$ we have

$$(\Gamma_1 \cdot \Gamma_2)(\omega) = \bigvee \{ \Delta(\omega) : \Delta \leq \Gamma_1 \cdot \Gamma_2 \},$$

where Δ denote as always simple random variables. Consider now a $\psi \in (\Gamma_1 \cdot \Gamma_2)(\omega)$. In the algebraic information algebra (I_Ψ, D) this means that $\psi \leq \bigvee \{ \Delta(\omega) : \Delta \in \Gamma_1 \cdot \Gamma_2 \}$. The supremum on the right hand side is over a directed set in I_Ψ . By compactness, there is therefore a $\Delta \leq \Gamma_1 \cdot \Gamma_2$ such that $\psi \leq \Delta(\omega)$. Now, $\Delta \leq \Gamma_1 \cdot \Gamma_2$ means by the definition of combination in the ideal completion \mathcal{R} that there is a $\Delta_1 \leq \Gamma_1$, $\Delta_1 \in \mathcal{R}_s$, and a $\Delta_2 \leq \Gamma_2$, $\Delta_2 \in \mathcal{R}_s$ such that $\Delta \leq \Delta_1 \cdot \Delta_2$. This implies that $\psi \leq (\Delta_1 \cdot \Delta_2)(\omega) = \Delta_1(\omega) \cdot \Delta_2(\omega)$, where $\Delta_1(\omega) \in \Gamma_1(\omega)$ and $\Delta_2(\omega) \in \Gamma_2(\omega)$. But this shows that $\psi \in \Gamma_1(\omega) \cdot \Gamma_2(\omega)$.

Conversely, consider an element $\psi \in \Gamma_1(\omega) \cdot \Gamma_2(\omega)$. By the definition of the join in I_Ψ this means that there are elements $\psi_1, \psi_2 \in \Psi$ such that $\psi \leq \psi_1 \cdot \psi_2$, where $\psi_1 \leq \Gamma_1(\omega)$ and $\psi_2 \leq \Gamma_2(\omega)$. Now, $\psi_1 \leq \Gamma_1(\omega)$ means that $\psi_1 \leq \bigvee \{ \Delta(\omega) : \Delta \leq \Gamma_1 \}$. As above, by compactness, there is a $\Delta_1 \leq \Gamma_1$ such that $\psi_1 \leq \Delta_1(\omega)$. Similarly, there is a $\Delta_2 \leq \Gamma_2$ such that $\psi_2 \leq \Delta_2(\omega)$. Thus, $\psi \leq \Delta_1(\omega) \cdot \Delta_2(\omega) = (\Delta_1 \cdot \Delta_2)(\omega)$. Further $\Delta_1 \cdot \Delta_2 \leq \Gamma_1 \cdot \Gamma_2$. This implies $\psi \in (\Gamma_1 \cdot \Gamma_2)(\omega)$, hence finally $(\Gamma_1 \cdot \Gamma_2)(\omega) = \Gamma_1(\omega) \cdot \Gamma_2(\omega)$.

(2) Assume next that $\psi \in (\epsilon_x(\Gamma))(\omega)$. As above, using the definition of the random mapping associated with $\epsilon_x(\Gamma)$, this implies that there is a $\Delta \leq \epsilon_x(\Gamma)$ such that $\psi \leq \Delta(\omega)$. By the definition of $\epsilon_x(\Gamma)$ and compactness there is a $\Delta' \leq \Gamma$ such that $\Delta \leq \epsilon_x(\Delta')$. This implies $\psi \leq (\epsilon_x(\Delta'))(\omega) = \epsilon_x(\Delta'(\omega))$, which, together with $\Delta'(\omega) \leq \Gamma(\omega)$ shows that $\psi \in \epsilon_x(\Gamma(\omega))$.

Conversely, assume $\psi \in \epsilon_x(\Gamma(\omega))$. Then $\psi \leq \epsilon_x(\phi)$ for some $\phi \in \Gamma(\omega)$. Again, as above, there is a $\Delta \leq \Gamma$ such that $\phi \leq \Delta(\omega)$. Therefore, we

conclude that $\psi \leq \epsilon_x(\Delta(\omega)) = (\epsilon_x(\Delta))(\omega)$ and $\epsilon_x(\Delta) \leq \epsilon_x(\Gamma)$. This implies that $\psi \in (\epsilon_x(\Gamma))(\omega)$, hence $(\epsilon_x(\Gamma))(\omega) = \epsilon_x(\Gamma(\omega))$. \square

According to this lemma we have a homomorphism between the algebra of generalised random variables and random mappings. In fact, it is an embedding, since $\Gamma_1(\omega) = \Gamma_2(\omega)$ for all $\omega \in \Omega$ implies $\Gamma_1 = \Gamma_2$.

The next lemma strengthens Lemma 9.2.

Lemma 9.3 *If $X \subseteq \mathcal{R}$ is a directed set, then*

$$(\bigvee_{\Gamma \in X} \Gamma)(\omega) = \bigvee_{\Gamma \in X} \Gamma(\omega).$$

Proof. If $\Gamma' \in X$, then $\Gamma' \leq \bigvee_{\Gamma \in X} \Gamma$, hence $\Gamma'(\omega) \leq (\bigvee_{\Gamma \in X} \Gamma)(\omega)$ and therefore

$$\bigvee_{\Gamma \in X} \Gamma(\omega) \leq (\bigvee_{\Gamma \in X} \Gamma)(\omega).$$

Conversely, consider $\psi \in \Psi$ such that $\psi \leq (\bigvee_{\Gamma \in X} \Gamma)(\omega)$. Since, according to (9.22),

$$(\bigvee_{\Gamma \in X} \Gamma)(\omega) = \bigvee \{ \Delta(\omega) : \Delta \leq \bigvee_{\Gamma \in X} \Gamma \}$$

we have $\psi \leq \Delta(\omega)$ for some simple random variable $\Delta \leq \bigvee_{\Gamma \in X} \Gamma$. Now, since X is a directed set, by compactness, there is a $\Gamma \in X$ such that $\Delta \leq \Gamma$, hence $\Delta(\omega) \leq \Gamma(\omega)$. It follows then that $\psi \leq \bigvee_{\Gamma \in X} \Gamma(\omega)$, which in turn implies

$$(\bigvee_{\Gamma \in X} \Gamma)(\omega) \leq \bigvee_{\Gamma \in X} \Gamma(\omega).$$

This concludes the proof of the lemma. \square

The theory of generalised random variables developed above may be presented in a similar way in the framework of *algebraic information algebras*. Here is a sketch of the approach:

Example 9.1 *Generalised Random Variables in Algebraic Algebras:* Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be a algebraic information algebra with finite elements Ψ_f . We assume that $(\Psi_f, D; \leq, \perp, \cdot, \epsilon)$ is a subalgebra of $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. Define then *simple random variables* Δ with finite elements from Ψ_f as values. They form an idempotent generalised information algebra $(\mathcal{R}_s, D \leq, \perp, \cdot, \epsilon)$ with combination and extraction defined point-wise.

Since the ideal completion $(I_{\Psi_f}, D; \leq, \perp, \cdot, \epsilon)$ of the information algebra $(\Psi_f, D; \leq, \perp, \cdot, \epsilon)$ is isomorph to the algebraic algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ (see Section 7.2), the theory above applies to the present case. Generalised random variables in a algebraic information algebra can thus be considered as random mappings with values in Ψ , defined as point-wise limits of simple random variables with finite elements as values.

\ominus

As before with random mappings, there is no guarantee that the support $s_\Gamma(\psi)$ of a generalised random variable Γ is measurable for every $\psi \in \Psi$. But of course we can extend the support function to all of Ψ by the allocation of probability as proposed above. However, we shall show later that the degrees of support $sp_\Gamma(\psi)$ of a generalised random variable Γ is in fact determined by the degrees of support of its approximating simple random random variables, see Chapter 11 (Corollary 11.1).

Information algebras are closed under *finite* combinations. But there are information algebras which are closed under *countable* combinations. In this section we consider such algebras and uncertain information relative to such algebras. Here follows the definition which will be used in the sequel:

Definition 9.1 σ -Information Algebra. *An idempotent generalised information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is called a σ -information algebra, if*

1. *Countable Combination: Ψ is closed under countable combinations (joins).*
2. *Continuity of Extraction: For every monotone sequence $\psi_1 \leq \psi_2 \leq \dots \in \Psi$, and for any $x \in D$, it holds that*

$$\epsilon_x\left(\bigvee_{i=1}^{\infty} \psi_i\right) = \bigvee_{i=1}^{\infty} \epsilon_x(\psi_i).$$

The second condition is a weaker version of the continuity of extraction in compact information algebras, compare Theorem 7.2.

There are many examples of σ -information algebras. First of all, any *continuous* or *algebraic* information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is a σ -information algebra: Since in these cases Ψ is a complete lattice it is surely closed under countable join. The continuity of extraction follows from Theorems 7.2 and 7.19, since a monotone sequence is a directed set.

Further important examples of σ -information algebras are minimal extensions of information algebras $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ which are closed under countable combination. Such extensions can be obtained using ideal completion. In order to do this, we need to introduce a new concept. Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be an idempotent generalised information algebra and $(I_\Psi, D; \leq, \perp, \cdot, \epsilon)$ its ideal completion. A subset S of I_Ψ is called σ -closed, if it is closed under *countable* combinations or joins. The intersection of any family of σ -closed sets is also σ -closed. Further the set I_Ψ itself is σ -closed. Therefore, for any subset $X \subseteq I_\Psi$ we may define the σ -closure $\sigma(X)$ as the intersection of all σ -closed sets containing X .

We are particularly interested in $\sigma(\Psi)$, the σ -closure of Ψ in I_Ψ . Note that here, as in the sequel, we identify as usual Ψ with its embedding $I(\Psi)$ under the mapping $\psi \mapsto \downarrow \psi$ for simplicity of notation. Also we shall write ψ , even if we operate within I_Ψ . The σ -closure of Ψ can be characterized as follows:

Theorem 9.3 *If $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is an idempotent generalised information algebra, then*

$$\sigma(\Psi) = \{I \in I_\Psi : I = \bigvee_{i=1}^{\infty} \psi_i, \psi_i \in \Psi\}. \quad (9.23)$$

Proof. Clearly, the set on the right hand side of equation (9.23) contains Ψ and is contained in $\sigma(\Psi)$. We claim that this set is itself σ -closed. In fact, consider a countable set I_j of elements of this set, such that

$$I_j = \bigvee_{i=1}^{\infty} \psi_{j,i}$$

with $\psi_{j,i} \in \Psi$. Define the set $J = \{(j, i) : j = 1, 2, \dots; i = 1, 2, \dots\}$ and the sets $J_j = \{(j, i) : i = 1, 2, \dots\}$ for $j = 1, 2, \dots$, and $K_i = \{(h, j) : 1 \leq h, j \leq i\}$ for $i = 1, 2, \dots$. Then we have

$$J = \bigcup_{j=1}^{\infty} J_j = \bigcup_{i=1}^{\infty} K_i.$$

By the laws of associativity in the complete lattice I_Ψ we obtain then

$$\begin{aligned} \bigvee_{j=1}^{\infty} I_j &= \bigvee_{j=1}^{\infty} \left(\bigvee_{(j,i) \in J_j} \psi_{j,i} \right) \\ &= \bigvee_{(j,i) \in J} \psi_{j,i} = \bigvee_{i=1}^{\infty} \left(\bigvee_{(h,j) \in K_i} \psi_{h,j} \right). \end{aligned}$$

But $\bigvee_{(h,j) \in K_i} \psi_{h,j} \in \Psi$ for $i = 1, 2, \dots$. Hence $\bigvee_{j=1}^{\infty} I_j$ belongs itself to the set on the right hand side of (9.23). This means that this set is indeed σ -closed. Since the set contains Ψ , it contains also $\sigma(\Psi)$, hence it equals $\sigma(\Psi)$. \square

Consider now a *monotone* sequence $\psi_1 \leq \psi_2 \leq \dots$ of elements of Ψ . Its supremum exists in I_Ψ and belongs in fact to $\sigma(\Psi)$. The sequence is furthermore a directed set. Therefore, by Theorem 7.2 join commutes with information extraction, this is expressed in the following theorem. It shows that *continuity of extraction* holds:

Theorem 9.4 *For a monotone sequence $\psi_1 \leq \psi_2 \leq \dots$ of elements of Ψ , and for any $x \in D$, we have in $\sigma(\Psi)$ that*

$$\epsilon_x \left(\bigvee_{i=1}^{\infty} \psi_i \right) = \bigvee_{i=1}^{\infty} \epsilon_x(\psi_i). \quad (9.24)$$

Theorem 9.4 shows in particular that $\sigma(\Psi)$ is closed under focussing. In fact, if ϕ_i is any sequence of elements of Ψ , and $I = \bigvee_{i=1}^{\infty} \phi_i$, then we may define $\psi_i = \bigvee_{k=1}^i \phi_k \in \Psi$, such that ψ_k for $k = 1, 2, \dots$ is a monotone sequence and $I = \bigvee_{i=1}^{\infty} \phi_i = \bigvee_{i=1}^{\infty} \psi_i$. So, for $I \in \sigma(\Psi)$ and any $x \in D$ by Theorem 9.4

$$\epsilon_x(I) = \bigvee_{i=1}^{\infty} \epsilon_x(\psi_i), \quad (9.25)$$

where $\epsilon_x(\psi_i) \in \Psi$ and hence $\epsilon_x(I) \in \sigma(\Psi)$ by Theorem 9.3. As a σ -closed set, $\sigma(\Psi)$ is closed under combination. Therefore $(\sigma(\Psi), D; \leq, \perp, \cdot, \epsilon)$ is itself an information algebra, a subalgebra of $(\mathcal{R}_{\Psi}, D; \leq, \perp, \cdot, \epsilon)$. Since it is closed under combination (i.e. join) of countable sets, contains 0 and 1, and satisfies condition (9.24) it is a σ -information algebra, the σ -algebra induced by $(\Psi, D; \leq, \perp, \cdot, \epsilon)$.

A particular and import case of such a construction is $(\sigma(\Psi_f), D; \leq, \perp, \cdot, \epsilon)$ in an *algebraic* information algebra. Due to the Representation Theorem 7.5, this can be reduced to the situation of ideal completion, described above.

Consider simple random variables as defined as in Section 9.1. We may define a random mapping $\Gamma : \Omega \rightarrow I_{\Psi}$ from a countable family of simple random variables Δ_i by

$$\Gamma(\omega) = \bigvee_{i=1}^{\infty} \Delta_i(\omega).$$

We call such a random mapping Γ a *random variable* in the algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. Note that its values are in the ideal completion I_{Ψ} of Ψ . In the case of an algebraic information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$, the values of the simple random variables are considered to be finite, that is to be in Ψ_f .

Let now \mathcal{R}_{σ} be the family of random variables in the algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$.

Lemma 9.4 *A random variable Γ is always the supremum of a monotone increasing sequence $\Delta_1 \leq \Delta_2 \leq \dots$ of simple random variables, such that for all $\omega \in \Omega$,*

$$\Gamma(\omega) = \bigvee_{i=1}^{\infty} \Delta_i(\omega).$$

Proof. If Γ is a random variable, then $\Gamma(\omega) = \bigvee_{i=1}^{\infty} \Delta'_i(\omega)$ for some sequence Δ'_i of simple random variables. Define

$$\Delta_i = \bigvee_{j=1}^i \Delta'_j.$$

Then each Δ_i is a simple random variable, $i = 1, 2, \dots$ and $\Delta_1 \leq \Delta_2 \leq \dots$. From $\Delta'_i \leq \Delta_i$, we conclude that $\Gamma(\omega) = \bigvee_{i=1}^{\infty} \Delta'_i(\omega) \leq \bigvee_{i=1}^{\infty} \Delta_i(\omega)$. On the other hand, $\Delta_i(\omega) \leq \Gamma(\omega)$, hence $\bigvee_{i=1}^{\infty} \Delta_i(\omega) \leq \Gamma(\omega)$, such that finally $\Gamma(\omega) = \bigvee_{i=1}^{\infty} \Delta_i(\omega)$. \square

Random variables are random mappings and as such can be combined and extracted point-wise in the ideal completion (I_{Ψ}, D) :

1. *Combination*: $(\Gamma_1 \cdot \Gamma_2)(\omega) = \Gamma_1(\omega) \cdot \Gamma_2(\omega)$,
2. *Extraction*: $\epsilon_x(\Gamma)(\omega) = \epsilon_x(\Gamma(\omega))$.

We have to verify that the resulting random mappings still belong to \mathcal{R}_{σ} , that is are *random variables*. So, let

$$\Gamma_1 = \bigvee_{i=1}^{\infty} \Delta_{1,i}, \quad \Gamma_2 = \bigvee_{i=1}^{\infty} \Delta_{2,i}.$$

Then we obtain, using associativity of the supremum

$$\begin{aligned} (\Gamma_1 \cdot \Gamma_2)(\omega) &= (\Gamma_1 \vee \Gamma_2)(\omega) \\ &= \Gamma_1(\omega) \vee \Gamma_2(\omega) = \left(\bigvee_{i=1}^{\infty} \Delta_{1,i}(\omega) \right) \vee \left(\bigvee_{i=1}^{\infty} \Delta_{2,i}(\omega) \right) \\ &= \bigvee_{i=1}^{\infty} (\Delta_{1,i}(\omega) \vee \Delta_{2,i}(\omega)) = \bigvee_{i=1}^{\infty} (\Delta_{1,i} \vee \Delta_{2,i})(\omega). \end{aligned}$$

Since $\Delta_{1,i} \vee \Delta_{2,i} \in \mathcal{R}_s$, this proves that $\Gamma_1 \vee \Gamma_2 \in \mathcal{R}_{\sigma}$. Note then that, as usual, $\Gamma_1 \leq \Gamma_2$ if and only if $\Gamma_1(\omega) \leq \Gamma_2(\omega)$ for all $\omega \in \Omega$, since random variables are random mappings.

Further, let

$$\Gamma(\omega) = \bigvee_{i=1}^{\infty} \Delta_i(\omega),$$

where Δ_i is an increasing sequence of simple random variables (see Lemma 9.4). Then, by the continuity of extraction,

$$\epsilon_x(\Gamma)(\omega) = \epsilon_x(\Gamma(\omega)) = \epsilon_x\left(\bigvee_{i=1}^{\infty} \Delta_i(\omega)\right) = \bigvee_{i=1}^{\infty} \epsilon_x(\Delta_i(\omega)) = \bigvee_{i=1}^{\infty} \epsilon_x(\Delta_i)(\omega).$$

Again, if Δ_i are simple random variables, then so are the $\epsilon_x(\Delta_i)$, therefore $\epsilon_x(\Gamma)$ is indeed a random variable.

We expect $(\mathcal{R}_{\sigma}, D; \leq, \perp, \cdot, \epsilon)$ to form an information algebra, even a σ -algebra. This is indeed true. We use the following lemma to prove this statement:

Lemma 9.5 *Assume $\Gamma_i \in \mathcal{R}_\sigma$ for $i = 1, 2, \dots$ to be random variables. Then $\bigvee_{i=1}^\infty \Gamma_i$ exists in the information algebra of random mappings into I_Ψ , and for all $\omega \in \Omega$,*

$$\left(\bigvee_{i=1}^\infty \Gamma_i \right) (\omega) = \bigvee_{i=1}^\infty \Gamma_i(\omega)$$

Proof. Consider the random mapping η defined by $\eta(\omega) = \bigvee_{i=1}^\infty \Gamma_i(\omega)$. Since $\Gamma_i(\omega) \leq \bigvee_{i=1}^\infty \Gamma_i(\omega)$, it follows that $\Gamma_i \leq \eta$, hence η is an upper bound of the random mappings Γ_i . If χ is another upper bound, then $\Gamma_i(\omega) \leq \chi(\omega)$, hence $\eta(\omega) = \bigvee_{i=1}^\infty \Gamma_i(\omega) \leq \chi(\omega)$, therefore $\eta \leq \chi$. Thus, η is the supremum of the random mappings Γ_i . \square

Theorem 9.5 *The system $(\mathcal{R}_\sigma, D; \leq, \perp, \cdot, \epsilon)$ of random variables in the idempotent generalised information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$, with combination and extraction defined point-wise as above forms a σ -information algebra.*

Proof. As we have seen above, \mathcal{R}_σ is closed under combination (join) and extraction. The bottom element, the mapping $U(\omega) = 1$ as well as the top element $N(\omega) = 0$ belong also to \mathcal{R}_σ . So $(\mathcal{R}_\sigma, D; \leq, \perp, \cdot, \epsilon)$ is a subalgebra of the algebra of random mappings $(\mathcal{R}_{I_\Psi}, D; \leq, \perp, \cdot, \epsilon)$, hence an information algebra.

We show that $(\mathcal{R}_\sigma, D; \leq, \perp, \cdot, \epsilon)$ is σ -closed, that is, if $\Gamma_i \in \mathcal{R}_\sigma$ for $i = 1, 2, \dots$, then $\bigvee_{i=1}^\infty \Gamma_i \in \mathcal{R}_\sigma$. Let

$$\Gamma_j(\omega) = \bigvee_{i=1}^\infty \Delta_{j,i}(\omega), \text{ for } j = 1, 2, \dots,$$

where $\Delta_{j,i}$ are simple random variables, and define the random mapping Γ , using Lemma 9.5, by

$$\Gamma(\omega) = \left(\bigvee_{i=1}^n \Gamma_i \right) (\omega) = \bigvee_{j=1}^\infty \Gamma_j(\omega) = \bigvee_{j=1}^\infty \left(\bigvee_{i=1}^\infty \Delta_{j,i}(\omega) \right).$$

As in the proof of Theorem 9.3 define the sets $J_j = \{(j, i) : i = 1, 2, \dots\}$ and $K_i = \{(h, j) : 1 \leq h, j \leq i\}$. Then, as there, we obtain

$$\Gamma(\omega) = \bigvee_{i=1}^\infty \left(\bigvee_{(h,j) \in K_i} \Delta_{h,j}(\omega) \right).$$

Since $\bigvee_{(h,j) \in K_i} \Delta_{h,j}(\omega)$ defines simple random variables, the random mapping Γ is indeed a random variable and \mathcal{R}_σ is closed under countable combination.

It remains to verify the continuity of extraction. Assume $\Gamma_1 \leq \Gamma_2 \leq \dots$ be a monotone sequence of random variables in \mathcal{R}_σ and $x \in D$. Then, the

continuity of extraction in $(\mathcal{R}_\sigma, D; \leq, \perp, \cdot, \epsilon)$ follows from this property in $(\sigma(\Psi), D; \leq, \perp, \cdot, \epsilon)$, using Lemma 9.5 and the continuity of extraction in \mathcal{R}_σ , as follows:

$$\begin{aligned} \epsilon_x\left(\bigvee_{i=1}^{\infty} \Gamma_i\right)(\omega) &= \epsilon_x\left(\left(\bigvee_{i=1}^{\infty} \Gamma_i\right)(\omega)\right) = \epsilon_x\left(\bigvee_{i=1}^{\infty} \Gamma_i(\omega)\right) = \bigvee_{i=1}^{\infty} \epsilon_x(\Gamma_i(\omega)) \\ &= \bigvee_{i=1}^{\infty} \epsilon_x(\Gamma_i)(\omega) = \left(\bigvee_{i=1}^{\infty} \epsilon_x(\Gamma_i)\right)(\omega). \end{aligned}$$

So, we see that $\epsilon_x(\bigvee_{i=1}^{\infty} \Gamma_i) = \bigvee_{i=1}^{\infty} \epsilon_x(\Gamma_i)$. This concludes the proof. \square

Certainly, $(\mathcal{R}_s, D; \leq, \perp, \cdot, \epsilon)$ is a subalgebra of $(\mathcal{R}_\sigma, D; \leq, \perp, \cdot, \epsilon)$. Within the algebra $(\mathcal{R}_\sigma, D; \leq, \perp, \cdot, \epsilon)$, each element of \mathcal{R}_σ is the supremum of the simple random variables it dominates as the following lemma shows.

Lemma 9.6 *Let $\Gamma \in \mathcal{R}_\sigma$, defined by*

$$\Gamma(\omega) = \bigvee_{i=1}^{\infty} \Delta_i(\omega).$$

Then, in the information algebra $(\mathcal{R}_\sigma, D; \leq, \perp, \cdot, \epsilon)$

$$\Gamma = \bigvee_{i=1}^{\infty} \Delta_i = \bigvee \{\Delta : \Delta \in \mathcal{R}_s, \Delta \leq \Gamma\}. \quad (9.26)$$

Proof. The first equality in (9.26) follows directly from the definition of Γ . Trivially, Γ is an upper bound of the set $\{\Delta : \Delta \leq \Gamma\}$. If Γ' is another upper bound of this set, then it is also an upper bound of the Δ_i , hence $\Gamma \leq \Gamma'$. Therefore, Γ is the least upper bound of the set $\{\Delta : \Delta \leq \Gamma\}$. \square

This lemma shows that a random variable is also generalised random variable.

We now take the σ -closure of \mathcal{R}_s in the algebraic information algebra $(\mathcal{R}, D; \leq, \perp, \cdot, \epsilon)$ of generalised random variables. According to Theorem 9.3, elements of $\sigma(\mathcal{R}_s)$ are defined as

$$\Gamma = \bigvee_{i=1}^{\infty} \Delta_i, \text{ with } \Delta_i \in \mathcal{R}_s, \forall i = 1, 2, \dots$$

Then $(\sigma(\mathcal{R}_s), D; \leq, \perp, \cdot, \epsilon)$ is a σ -information algebra, containing \mathcal{R}_s , i.e. the simple random variables. To Γ we associate a random mapping, just as with generalised random variables, defined by

$$\Gamma(\omega) = \bigvee_{i=1}^{\infty} \Delta_i(\omega), \text{ with } \Delta_i \in \mathcal{R}_s, \forall i = 1, 2, \dots$$

Note that $\Gamma(\omega) \in \sigma(\Psi)$ by Theorem 9.3. Therefore, the elements of $\sigma(\mathcal{R}_s)$ are *random variables* with values in the information algebra $(\sigma(\Psi), D; \leq, \perp, \cdot, \epsilon)$. This shows the equivalence of taking the σ -closure of \mathcal{R}_s and the definition of random variables as suprema of sequences of simple random variables.

Chapter 10

Allocations of Probability

10.1 Algebra of Allocations of Probability

In Section 9.2 we have introduced the concept of an *allocation of probability* (a.o.p) as a means to extend the degrees of support of a random mapping beyond the measurable elements ϕ , that is, the elements for which $s_\Gamma(\phi) \in \mathcal{A}$. These allocations of probability play an important role in the theory of uncertain information. Therefore, we start here with a study of this concept, first independently of its relation to random mappings and random variables. In the subsequent Section 10.2 we examine the relation between random mappings and their associated allocations of probability.

Random mappings, and in particular generalised random variables and random variables, provide means to model explicitly the mechanisms which generate uncertain information. We refer to (Kohlas & Monney, 1995; Haenni *et al.*, 2000; Kohlas, 2003a; Kohlas & Monney, 2007; Pouly & Kohlas, 2011) for more specific applications of this idea. Alternatively, allocations of probability may serve to directly assign beliefs to pieces of information. This is more in the spirit of a subjective, epistemological description of belief, advocated especially by G. Shafer (Shafer, 1973; Shafer, 1976; Shafer, 1979). In this view, allocations of probability are taken as the primitive elements, rather than random variables or hints. This is the point of view developed in this section (see also (Kohlas, 1997; Kohlas, 2003b)).

We introduce the concept of an allocation of probability:

Definition 10.1 *Allocation of Probability.* *If $(\Psi; \leq)$ is a bounded join-semilattice and (μ, \mathcal{B}) a probability algebra, then an allocation of probability (a.o.p) is a mapping $\rho : \Psi \rightarrow \mathcal{B}$ such that*

$$(A1) \quad \rho(1) = \top,$$

$$(A2) \quad \rho(\phi \vee \psi) = \rho(\phi) \wedge \rho(\psi).$$

If furthermore $\rho(0) = \perp$ holds, then the allocation is called normalised .

We shall apply this definition especially to idempotent generalised information algebras $(\Psi, D; \leq, \perp, \cdot, \epsilon)$, where in the semilattice (Ψ, \leq) join corresponds to combination. (A1) says then that the full belief is allocated to the trivial vacuous information. More important is (A2). It says that the belief allocated to a combined information $\phi \cdot \psi$ equals the *common* part of belief $\rho(\phi) \wedge \rho(\psi)$ allocated to both of the two pieces of information ϕ and ψ individually. We remind that the a.o.p derived from a random mapping satisfies these two properties (see (9.11)). Note, that if $\phi \leq \psi$, that is, $\phi \vee \psi = \psi$, then $\rho(\phi \vee \psi) = \rho(\phi) \wedge \rho(\psi) = \rho(\psi)$, hence $\rho(\psi) \leq \rho(\phi)$. A particular a.o.p is defined by $\nu(\phi) = \perp$, unless $\phi = 1$, in which case $\nu(1) = \top$. This is called the *vacuous allocation*; no belief is allocated to a non-trivial piece of information. It is associated with the vacuous information represented by the random mapping $\Gamma(\omega) = 1$ for all $\omega \in \Omega$.

We may think of an allocation of probability as the description of a body of belief relative to pieces of information in an information algebra $((\Psi, D; \leq, \perp, \cdot, \epsilon))$ obtained from a source of information. Two (or more) distinct sources of information will lead to the definition of two (or more) corresponding allocations of probability. Thus, in a general setting, let A_Ψ be the set of all allocations of probability in Ψ in (\mathcal{B}, μ) . Select two allocations $\rho_i, i = 1, 2$, from A_Ψ . How can they be combined in order to synthesise the two bodies of information they represent into a single, aggregated body?

The basic idea is as follows: Consider a piece of information ψ in Ψ . If now ψ_1 and ψ_2 are two other pieces of information in Ψ , such that $\psi \leq \psi_1 \cdot \psi_2$, then the common belief $\rho_1(\psi_1) \wedge \rho_2(\psi_2)$ allocated to ψ_1 and to ψ_2 by the two allocations ρ_1 and ρ_2 respectively, is a belief allocated to ψ by the two allocations simultaneously. That is, the total belief $\rho(\psi)$ to be allocated to ψ by the two allocations ρ_1 and ρ_2 together must equal at least the *common* belief allocated to ψ_1 and ψ_2 individually by each of the two allocations respectively, that is

$$\rho(\psi) \geq \rho_1(\psi_1) \wedge \rho_2(\psi_2). \quad (10.1)$$

In the absence of other information, it seems then reasonable to define the combined belief in ψ , as obtained from the two sources of information, as the least upper bound of all these implied beliefs,

$$\rho(\psi) = \bigvee \{ \rho_1(\psi_1) \wedge \rho_2(\psi_2) : \psi \leq \psi_1 \cdot \psi_2 \}. \quad (10.2)$$

This defines indeed a new allocation of probability:

Theorem 10.1 *Let $\rho_1, \rho_2 \in A_\Psi$ be two allocations of probability. The map $\rho : \Psi \rightarrow \mathcal{B}$ as defined by (10.2) is then an allocation of probability.*

Proof. First, we have

$$\begin{aligned} \rho(1) &= \bigvee \{ \rho_1(\psi_1) \wedge \rho_2(\psi_2) : 1 \leq \psi_1 \cdot \psi_2 \} \\ &= \rho_1(1) \wedge \rho_2(1) = \top. \end{aligned}$$

So (A1) is satisfied.

Next, let $\psi_1, \psi_2 \in \Psi$. By definition we have

$$\rho(\psi_1 \vee \psi_2) = \bigvee \{\rho_1(\phi_1) \wedge \rho_2(\phi_2) : \psi_1 \cdot \psi_2 \leq \phi_1 \cdot \phi_2\}.$$

Now, $\psi_1 \leq \psi_1 \vee \psi_2$ implies that

$$\begin{aligned} & \bigvee \{\rho_1(\phi_1) \wedge \rho_2(\phi_2) : \psi_1 \cdot \psi_2 \leq \phi_1 \cdot \phi_2\} \\ & \leq \bigvee \{\rho_1(\phi_1) \wedge \rho_2(\phi_2) : \psi_1 \leq \phi_1 \cdot \phi_2\} = \rho_1(\psi_1) \end{aligned}$$

and similarly for ψ_2 . Thus, we have $\rho(\psi_1 \vee \psi_2) \leq \rho(\psi_1), \rho(\psi_2)$, that is $\rho(\psi_1 \vee \psi_2) \leq \rho(\psi_1) \wedge \rho(\psi_2)$.

On the other hand,

$$\begin{aligned} & \{(\phi_1, \phi_2) : \psi_1 \vee \psi_2 \leq \phi_1 \cdot \phi_2\} \\ & \supseteq \{(\phi_1, \phi_2) : \phi_1 = \phi'_1 \cdot \phi''_1, \phi_2 = \phi'_2 \cdot \phi''_2, \psi_1 \leq \phi'_1 \cdot \phi'_2, \psi_2 \leq \phi''_1 \cdot \phi''_2\}. \end{aligned}$$

By the distributive law for complete Boolean algebras we obtain then

$$\begin{aligned} & \rho(\psi_1 \vee \psi_2) \\ & \geq \bigvee \{\rho_1(\phi'_1 \cdot \phi''_1) \wedge \rho_2(\phi'_2 \cdot \phi''_2) : \psi_1 \leq \phi'_1 \cdot \phi'_2, \psi_2 \leq \phi''_1 \cdot \phi''_2\} \\ & = \bigvee \{(\rho_1(\phi'_1) \wedge \rho_1(\phi''_1)) \wedge (\rho_2(\phi'_2) \wedge \rho_2(\phi''_2)) : \psi_1 \leq \phi'_1 \cdot \phi'_2, \psi_2 \leq \phi''_1 \cdot \phi''_2\} \\ & = \left(\bigvee \{\rho_1(\phi'_1) \wedge \rho_2(\phi'_2) : \psi_1 \leq \phi'_1 \cdot \phi'_2\} \right) \wedge \\ & \quad \left(\bigvee \{\rho_1(\phi''_1) \wedge \rho_2(\phi''_2) : \psi_2 \leq \phi''_1 \cdot \phi''_2\} \right) \\ & = \rho(\psi_1) \wedge \rho(\psi_2). \end{aligned} \tag{10.3}$$

This implies finally that $\rho(\psi_1 \vee \psi_2) = \rho(\psi_1) \wedge \rho(\psi_2)$. Thus (A2) holds too and ρ is indeed an allocation of probability. \square

In this way, in the set of allocations of probability A_Ψ a binary combination operation is defined. We denote this operation by \cdot . Thus, ρ as defined by (10.2) is written as $\rho = \rho_1 \cdot \rho_2$. The following theorem gives us the elementary properties of this operation.

Theorem 10.2 *The combination operation, as defined by (10.2), is commutative, associative, idempotent and the vacuous allocation is the neutral element of this operation.*

Proof. The commutativity of (10.2) is evident. For the associativity note that for a $\psi \in \Psi$ we have, due to the associativity and distributivity of meet and join in complete Boolean algebras,

$$\begin{aligned} & ((\rho_1 \cdot \rho_2) \cdot \rho_3)(\psi) \\ & = \bigvee \{(\rho_1 \cdot \rho_2)(\phi_{1,2}) \wedge \rho_3(\phi_3) : \psi \leq \phi_{1,2} \cdot \phi_3\} \\ & = \bigvee \{ \bigvee \{\rho_1(\phi_1) \wedge \rho_2(\phi_2) : \phi_{1,2} \leq \phi_1 \cdot \phi_2\} \wedge \rho_3(\phi_3) : \psi \leq \phi_{1,2} \cdot \phi_3\} \\ & = \bigvee \{\rho_1(\phi_1) \wedge \rho_2(\phi_2) \wedge \rho_3(\phi_3) : \psi \leq \phi_1 \cdot \phi_2 \cdot \phi_3\}. \end{aligned}$$

For $(\rho_1 \cdot (\rho_2 \cdot \rho_3))(\phi)$ we obtain exactly the same result in the same way. This proves associativity.

To show idempotency consider

$$\begin{aligned} (\rho \cdot \rho)(\psi) &= \bigvee \{\rho(\phi_1) \wedge \rho(\phi_2) : \psi \leq \phi_1 \cdot \phi_2\} \\ &= \bigvee \{\rho(\phi_1 \cdot \phi_2) : \psi \leq \phi_1 \cdot \phi_2\} = \rho(\psi) \end{aligned}$$

since the last supremum is attained for $\phi_1 = \phi_2 = \psi$.

Finally let ν denote the vacuous allocation. Then, for any allocation ρ and any $\psi \in \Psi$ we have, noting that $\nu(\phi) = \perp$, unless $\phi = 1$, in which case $\nu(1) = \top$,

$$(\rho \cdot \nu)(\psi) = \bigvee \{\rho(\phi_1) \wedge \nu(\phi_2) : \psi \leq \phi_1 \cdot \phi_2\} = \rho(\psi).$$

This shows that ν is the neutral element for combination. \square

This theorem shows that A_Ψ is a *semilattice*. Indeed, a partial order between allocations can be introduced as usual by defining $\rho_1 \leq \rho_2$ if $\rho_1 \cdot \rho_2 = \rho_2$. This means that for all $\psi \in \Psi$,

$$\rho_1 \cdot \rho_2(\psi) = \bigvee \{\rho_1(\psi_1) \wedge \rho_2(\psi_2) : \psi \leq \psi_1 \cdot \psi_2\} = \rho_2(\psi).$$

We have therefore always $\rho_1(\psi_1) \wedge \rho_2(\psi_2) \leq \rho_2(\psi)$ if $\psi \leq \psi_1 \cdot \psi_2$. Take now $\psi_1 = \psi$ and $\psi_2 = 1$, such that $\psi \leq \psi \cdot 1 = \psi$, to obtain $\rho_1(\psi) \wedge \rho_2(1) = \rho_1(\psi) \leq \rho_2(\psi)$. Thus we have $\rho_1 \leq \rho_2$ if and only if $\rho_1(\psi) \leq \rho_2(\psi)$ for all $\psi \in \Psi$. Clearly, the combination $\rho_1 \cdot \rho_2$ is the supremum of the two a.o.p in this order. Therefore we shall henceforth write $\rho_1 \vee \rho_2$ for this combination if we want to emphasise the order-theoretic aspects. The vacuous a.o.p is the bottom element of this semilattice. And the a.o.p defined by $\zeta(\psi) = \top$ for all information elements is the top element to the semilattice A_Ψ , so that $\rho \vee \zeta = \zeta$. So the semilattice of a.o.ps A_Ψ is a bounded semilattice.

Next we turn to the operation of extracting a part of an allocation of probability according to an operator ϵ_x . Let ρ be an allocation of probability on an information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ and $x \in D$. Just as it is possible to extract a part of a piece of information ψ from Ψ with the aid of the operator ϵ_x , it should also be possible to focus the belief represented by the a.o.p ρ to the information supported by the domain x . This means to extract the information related to x from ρ . Thus, for a $\psi \in \Psi$ consider the beliefs allocated to pieces of information ϕ which are supported by x and which entail ψ , i.e. $\psi \leq \phi = \epsilon_x(\phi)$. The part of the belief allocated to ψ and relating to the domain x , $\epsilon_x(\rho)(\psi)$ must then be at least $\rho(\phi)$,

$$\epsilon_x(\rho)(\psi) \geq \rho(\phi) \text{ for any } \phi = \epsilon_x(\phi) \geq \psi. \quad (10.4)$$

In the absence of other information, it seems again, as above, reasonable to define $\epsilon_x(\rho)(\psi)$ to be the least upper bound of all these implied supports,

$$\epsilon_x(\rho)(\psi) = \bigvee \{\rho(\phi) : \psi \leq \phi = \epsilon_x(\phi)\}. \quad (10.5)$$

This defines indeed an allocation of probability:

Theorem 10.3 *Let $\rho \in A_\Psi$ be an allocation of probability. The map $\epsilon_x(\rho) : \Psi \rightarrow \mathcal{B}$ as defined by (10.5) is an allocation of probability.*

Proof. We have by definition

$$\epsilon_x(\rho)(1) = \bigvee \{\rho(\phi) : 1 \leq \phi = \epsilon_x(\phi)\} = \rho(1) = \top.$$

Thus (A1) is verified.

Again by definition,

$$\epsilon_x(\rho)(\phi_1 \cdot \phi_2) = \bigvee \{\rho(\phi) : \phi_1 \cdot \phi_2 \leq \phi = \epsilon_x(\phi)\}.$$

From $\phi_1, \phi_2 \leq \phi_1 \cdot \phi_2$ it follows that $\epsilon_x(\rho)(\phi_1 \vee \phi_2) \leq \epsilon_x(\rho)(\phi_1), \epsilon_x(\rho)(\phi_2)$ and thus $\epsilon_x(\rho)(\phi_1 \cdot \phi_2) \leq \epsilon_x(\rho)(\phi_1) \wedge \epsilon_x(\rho)(\phi_2)$.

On the other hand, we have

$$\begin{aligned} & \{\psi : \phi_1 \cdot \phi_2 \leq \psi = \epsilon_x(\psi)\} \\ & \supseteq \{\psi = \psi_1 \cdot \psi_2 : \phi_1 \leq \psi_1 = \epsilon_x(\psi_1), \phi_2 \leq \psi_2 = \epsilon_x(\psi_2)\}. \end{aligned}$$

From this we obtain, using the distributive law for complete Boolean algebras,

$$\begin{aligned} & \epsilon_x(\rho)(\phi_1 \cdot \phi_2) \\ & \geq \bigvee \{\rho(\psi_1 \cdot \psi_2) : \phi_1 \leq \psi_1 = \epsilon_x(\psi_1), \phi_2 \leq \psi_2 = \epsilon_x(\psi_2)\} \\ & = \bigvee \{\rho(\psi_1) \wedge \rho(\psi_2) : \phi_1 \leq \psi_1 = \epsilon_x(\psi_1), \phi_2 \leq \psi_2 = \epsilon_x(\psi_2)\} \\ & = \left(\bigvee \{\rho(\psi_1) : \phi_1 \leq \psi_1 = \epsilon_x(\psi_1)\} \right) \wedge \left(\bigvee \{\rho(\psi_2) : \phi_2 \leq \psi_2 = \epsilon_x(\psi_2)\} \right) \\ & = \rho(\phi_1) \wedge \rho(\phi_2). \end{aligned}$$

This proves property (A2) for an allocation of support. \square

We are now going to show that the a.o.p in A_Ψ in fact define an idempotent, generalised information algebra $(A_\Psi, D; \leq, \perp, \cdot, \epsilon)$, without the Support Axiom A2 (unless (D, \leq) has a largest element). The Quasi-Separoid Axiom A0 is inherited from the underlying algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. The Semigroup Axiom A1 is proved in Theorem 10.2. Concerning the Unit and Null Axiom A3 we have already noted above that the vacuous allocation ν is the unit element of combination and the a.o.p ζ is the null element of combination. It remains to verify that $\epsilon_x(\rho) = \zeta$ implies $\rho = \zeta$. This holds, since $\epsilon_x(\rho) = \zeta$ means in particular $\epsilon_x(\rho)(1) = \top$, which implies $\rho(1) = \top$, hence indeed $\rho = \zeta$. The Extraction and Combination Axioms A4 and A5 are proved in the following two theorems.

Theorem 10.4 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be an idempotent generalised information algebra and $\rho \in A_\Psi$ an allocation of probability with support x , i.e. $\epsilon_x(\rho) = \rho$. Then $x \perp y | z$ implies $\epsilon_y(\rho) = \epsilon_y(\epsilon_z(\rho))$.*

Proof. Consider an element $\psi \in \Psi$. Then

$$\epsilon_y(\rho)(\psi) = \bigvee \{\rho(\phi) : \psi \leq \phi = \epsilon_y(\phi)\}.$$

Similarly, we obtain, using associativity of supremum,

$$\begin{aligned} \epsilon_y(\epsilon_z(\rho))(\psi) &= \bigvee \{\epsilon_z(\rho)(\psi') : \psi \leq \psi' = \epsilon_y(\psi')\} \\ &= \bigvee \{ \bigvee \{\rho(\phi) : \psi' \leq \phi = \epsilon_z(\phi)\} : \psi \leq \psi' = \epsilon_y(\psi') \} \\ &= \bigvee \{\rho(\phi) : \psi \leq \psi' = \epsilon_y(\psi') \leq \phi = \epsilon_z(\phi)\}. \end{aligned}$$

Using the assumption that $\rho = \epsilon_x(\rho)$, we obtain further

$$\begin{aligned} \epsilon_y(\rho)(\psi) &= \bigvee \{ \bigvee \{\rho(\phi') : \phi \leq \phi' = \epsilon_x(\phi')\} : \psi \leq \phi = \epsilon_y(\phi) \} \\ &= \bigvee \{\rho(\phi') : \psi \leq \phi = \epsilon_y(\phi) \leq \phi' = \epsilon_x(\phi')\}. \end{aligned}$$

and

$$\begin{aligned} \epsilon_y(\epsilon_z(\rho))(\psi) &= \bigvee \{ \bigvee \{\rho(\phi') : \phi \leq \phi' = \epsilon_x(\phi')\} : \psi \leq \psi' = \epsilon_y(\psi') \leq \phi = \epsilon_z(\phi) \} \\ &= \bigvee \{\rho(\phi') : \psi \leq \psi' = \epsilon_y(\psi') \leq \phi = \epsilon_z(\phi) \leq \phi' = \epsilon_x(\phi')\}. \end{aligned}$$

Define

$$\begin{aligned} A &= \{ \phi' : \exists \phi \text{ such that } \psi \leq \phi = \epsilon_y(\phi) \leq \phi' = \epsilon_x(\phi') \} \\ B &= \{ \phi' : \exists \phi, \psi' \text{ such that } \psi \leq \psi' = \epsilon_y(\psi') \leq \phi = \epsilon_z(\phi) \leq \phi' = \epsilon_x(\phi') \}. \end{aligned}$$

Note that both A and B depend on ψ . Then we have

$$\epsilon_y(\rho)(\psi) = \bigvee_{\phi' \in A} \rho(\phi'), \quad \epsilon_y(\epsilon_z(\rho))(\psi) = \bigvee_{\phi' \in B} \rho(\phi').$$

We claim that $A = B$. In fact, consider $\phi' \in A$ and let ϕ be such that $\psi \leq \phi = \epsilon_y(\phi) \leq \phi'$. Note that $x \perp y | z$ implies that $\epsilon_y(\phi') = \epsilon_y(\epsilon_z(\phi'))$. Then take $\psi' = \phi$ and $\phi'' = \epsilon_z(\phi')$. It follows that $\psi \leq \psi' = \epsilon_y(\psi') \leq \phi'' = \epsilon_z(\phi'') \leq \phi' = \epsilon_x(\phi')$. But this means that $\phi' \in B$. Conversely, if $\phi' \in B$, let ψ' be such that $\psi \leq \psi' = \epsilon_y(\psi') \leq \phi = \epsilon_z(\phi) \leq \phi'$. then take $\phi'' = \psi'$. It follows that $\psi \leq \phi'' = \epsilon_y(\phi'') \leq \phi' = \epsilon_x(\phi')$ and hence $\phi' \in A$. So $A = B$ and $\epsilon_y(\rho) = \epsilon_y(\epsilon_z(\rho))$. \square

Combination Axiom A5 holds also for a.o.p

Theorem 10.5 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be an idempotent generalised information algebra and $\rho_1, \rho_2 \in A_\Psi$ two allocations of probability with supports x and y respectively. Then $x \perp y | z$ implies $\epsilon_z(\rho_1 \cdot \rho_2) = \epsilon_z(\rho_1) \cdot \epsilon_z(\rho_2)$.*

Proof. The proof goes as the proof of Theorem 10.4: We first expand for an element $\psi \in \Psi$ the terms $\epsilon_z(\rho_1 \cdot \rho_2)(\psi)$ and $(\epsilon_z(\rho_1) \cdot \epsilon_z(\rho_2))(\psi)$ according to their definitions, using the associative and distributive laws in a complete Boolean algebra:

$$\begin{aligned}
& \epsilon_z(\rho_1 \cdot \rho_2)(\psi) \\
&= \bigvee \{ (\rho_1 \cdot \rho_2)(\phi) : \psi \leq \phi = \epsilon_z(\phi) \} \\
&= \bigvee \{ \bigvee \{ \rho_1(\phi_1) \wedge \rho_2(\phi_2) : \phi \leq \phi_1 \cdot \phi_2 \} : \psi \leq \phi = \epsilon_z(\phi) \} \\
&= \bigvee \{ \rho_1(\phi_1) \wedge \rho_2(\phi_2) : \psi \leq \phi = \epsilon_z(\phi) \leq \phi_1 \cdot \phi_2 \}, \\
\\
& (\epsilon_z(\rho_1) \cdot \epsilon_z(\rho_2))(\psi) \\
&= \bigvee \{ \epsilon_z(\rho_1)(\phi_1) \wedge \epsilon_z(\rho_2)(\phi_2) : \psi \leq \phi_1 \cdot \phi_2 \} \\
&= \bigvee \{ \left(\bigvee \{ \rho_1(\psi'_1) : \phi_1 \leq \psi'_1 = \epsilon_z(\psi'_1) \} \right) \\
&\quad \wedge \bigvee \{ \left(\bigvee \{ \rho_2(\psi'_2) : \phi_2 \leq \psi'_2 = \epsilon_z(\psi'_2) \} \right) : \psi \leq \phi_1 \cdot \phi_2 \} \\
&= \bigvee \{ \rho_1(\psi'_1) \wedge \rho_2(\psi'_2) : \psi \leq \phi_1 \cdot \phi_2, \\
&\quad \phi_1 \leq \psi'_1 = \epsilon_z(\psi'_1), \phi_2 \leq \psi'_2 = \epsilon_z(\psi'_2) \}.
\end{aligned}$$

Now, introduce in both expansions the assumptions $\rho_1 = \epsilon_x(\rho_1)$ and $\rho_2 = \epsilon_y(\rho_2)$, which yields

$$\begin{aligned}
& \epsilon_z(\rho_1 \cdot \rho_2)(\psi) \\
&= \bigvee \{ \left(\bigvee \{ \rho_1(\phi'_1) : \phi_1 \leq \phi'_1 = \epsilon_x(\phi'_1) \} \right) \\
&\quad \wedge \left(\bigvee \{ \rho_2(\phi'_2) : \phi_2 \leq \phi'_2 = \epsilon_y(\phi'_2) \} \right) : \psi \leq \phi = \epsilon_z(\phi) \leq \phi_1 \cdot \phi_2 \} \\
&= \bigvee \{ \bigvee \{ \rho_1(\phi'_1) \wedge \rho_2(\phi'_2) : \phi_1 \leq \phi'_1 = \epsilon_x(\phi'_1), \phi_2 \leq \phi'_2 = \epsilon_y(\phi'_2) \} : \\
&\quad \psi \leq \phi = \epsilon_z(\phi) \leq \phi_1 \cdot \phi_2 \} \\
&= \bigvee \{ \rho_1(\phi'_1) \wedge \rho_2(\phi'_2) : \psi \leq \phi = \epsilon_z(\phi) \leq \phi_1 \cdot \phi_2, \phi_1 \leq \phi'_1 = \epsilon_x(\phi'_1), \\
&\quad \phi_2 \leq \phi'_2 = \epsilon_y(\phi'_2) \},
\end{aligned}$$

$$\begin{aligned}
& (\epsilon_z(\rho_1) \cdot \epsilon_z(\rho_2))(\psi) \\
&= \bigvee \{ \left(\bigvee \{ \rho_1(\phi'_1) : \psi'_1 \leq \phi'_1 = \epsilon_x(\phi'_1) \} \right) \wedge \left(\bigvee \{ \rho_2(\phi'_2) : \psi'_2 \leq \phi'_2 = \epsilon_y(\phi'_2) \} \right) : \\
&\quad \psi \leq \phi_1 \cdot \phi_2, \phi_1 \leq \psi'_1 = \epsilon_z(\psi'_1), \phi_2 \leq \psi'_2 = \epsilon_z(\psi'_2) \} \\
&= \bigvee \{ \rho_1(\phi'_1) \wedge \rho_2(\phi'_2) : \psi \leq \phi_1 \cdot \phi_2,
\end{aligned}$$

$$\phi_1 \leq \psi'_1 = \epsilon_z(\psi'_1) \leq \phi'_1 = \epsilon_x(\phi'_1), \phi_2 \leq \psi'_2 = \epsilon_z(\psi'_2) \leq \phi'_2 = \epsilon_y(\phi'_2)\}.$$

According to these expansion, we define the sets

$$\begin{aligned} A &= \{(\phi'_1, \phi'_2) : \exists \phi, \phi_1, \phi_2 \text{ such that } \psi \leq \phi = \epsilon_z(\phi) \leq \phi_1 \cdot \phi_2, \\ &\quad \phi_1 \leq \phi'_1 = \epsilon_x(\phi'_1), \phi_2 \leq \phi'_2 = \epsilon_y(\phi'_2)\}, \\ B &= \{(\phi'_1, \phi'_2) : \exists \phi_1, \phi_2, \psi'_1, \psi'_2 \text{ such that } \psi \leq \phi_1 \cdot \phi_2, \\ &\quad \phi_1 \leq \psi'_1 = \epsilon_z(\psi'_1) \leq \phi'_1 = \epsilon_x(\phi'_1), \phi_2 \leq \psi'_2 = \epsilon_z(\psi'_2) \leq \phi'_2 = \epsilon_y(\phi'_2)\}. \end{aligned}$$

We remark again, that both A and B depend on ψ and

$$\begin{aligned} \epsilon_z(\rho_1 \cdot \rho_2)(\psi) &= \bigvee_{(\phi'_1, \phi'_2) \in A} \rho_1(\phi'_1) \wedge \rho_2(\phi'_2), \\ \epsilon_z(\rho_1) \cdot \epsilon_z(\rho_2)(\psi) &= \bigvee_{(\phi'_1, \phi'_2) \in B} \rho_1(\phi'_1) \wedge \rho_2(\phi'_2). \end{aligned}$$

As before, we claim that $A = B$. Suppose $(\phi'_1, \phi'_2) \in A$ so that there are elements ϕ, ϕ'_1, ϕ'_2 such that $\psi \leq \phi = \epsilon_z(\phi) \leq \phi_1 \cdot \phi_2$ and $\phi_1 \leq \phi'_1 = \epsilon_x(\phi'_1), \phi_2 \leq \phi'_2 = \epsilon_y(\phi'_2)$. Define then $\phi''_1 = \psi'_1 = \epsilon_z(\phi'_1)$ and $\phi''_2 = \psi'_2 = \epsilon_z(\phi'_2)$. Then we have $\epsilon_z(\phi) \leq \epsilon_z(\phi'_1 \cdot \phi'_2)$. But the assumption $x \perp y | z$ implies $\epsilon_z(\phi'_1 \cdot \phi'_2) = \epsilon_z(\phi'_1) \cdot \epsilon_z(\phi'_2)$, hence we obtain from $\psi \leq \epsilon_z(\phi)$ that $\psi \leq \phi''_1 \cdot \phi''_2$. We have further $\phi''_1 \leq \psi'_1 = \epsilon_z(\psi'_1) \leq \phi'_1 = \epsilon_x(\phi'_1)$ and $\phi''_2 \leq \psi'_2 = \epsilon_z(\psi'_2) \leq \phi'_2 = \epsilon_y(\phi'_2)$. This shows that $(\phi'_1, \phi'_2) \in B$.

Conversely, consider $(\phi'_1, \phi'_2) \in B$ such that there are elements ψ'_1, ψ'_2 and ϕ_1, ϕ_2 satisfying $\psi \leq \phi_1 \cdot \phi_2$ and $\phi_1 \leq \psi'_1 = \epsilon_z(\psi'_1) \leq \phi'_1 = \epsilon_x(\phi'_1)$ and $\phi_2 \leq \psi'_2 = \epsilon_z(\psi'_2) \leq \phi'_2 = \epsilon_y(\phi'_2)$. From $\psi'_1 = \epsilon_z(\psi'_1) \leq \phi'_1$ it follows that $\epsilon_z(\psi'_1) \leq \epsilon_z(\phi'_1)$, and, similarly, $\epsilon_z(\psi'_2) \leq \epsilon_z(\phi'_2)$. Define $\phi = \epsilon_z(\psi'_1) \cdot \epsilon_z(\psi'_2)$. Then, we have that $\psi \leq \phi = \epsilon_z(\phi) = \epsilon_z(\psi'_1) \cdot \epsilon_z(\psi'_2) \leq \epsilon_z(\phi'_1) \cdot \epsilon_z(\phi'_2)$. Define now further $\phi''_1 = \epsilon_z(\phi'_1)$ and $\phi''_2 = \epsilon_z(\phi'_2)$. Then, we obtain $\psi \leq \phi = \epsilon_z(\phi) \leq \phi''_1 \cdot \phi''_2$ and $\phi''_1 \leq \phi'_1 = \epsilon_x(\phi'_1), \phi''_2 \leq \phi'_2 = \epsilon_y(\phi'_2)$. This shows that $(\phi'_1, \phi'_2) \in A$, hence $A = B$ and therefore $\epsilon_z(\rho_1 \cdot \rho_2) = \epsilon_z(\rho_1) \cdot \epsilon_z(\rho_2)$. \square

We have already noted above, that combination is idempotent, that is, $\rho \cdot \rho = \rho$. However we need more, namely $\rho \cdot \epsilon_x(\rho) = \rho$, which is the Idempotency Axiom A5. In fact, we have obviously $\epsilon_x(\rho)(\psi) \leq \rho(\psi)$ for all $\psi \in \Psi$, hence $\epsilon_x(\rho) \leq \rho$ and therefore $\rho \cdot \epsilon_x(\rho) = \rho$. All this together proves that the a.o.p form an idempotent generalised information algebra.

Theorem 10.6 *If $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is an idempotent generalised information algebra, then $(A_\Psi, D; \leq, \perp, \cdot, \epsilon)$ is also an idempotent generalised information algebra.*

We show now that the algebra $(A_\Psi, D; \leq, \perp, \cdot, \epsilon)$ is in fact an extension of the information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. Consider for any $\phi \in \Psi$ the following map of Ψ into \mathcal{B} :

$$\rho_\phi(\psi) = \begin{cases} \top & \text{if } \psi \leq \phi, \\ \perp & \text{otherwise,} \end{cases} \quad (10.6)$$

It allocates total belief to all elements of information implied by ϕ , that is to all elements of the principal ideal $\downarrow\phi$, and no belief to all other elements. This map is clearly an *allocation of probability*; it is called a *deterministic allocation*. It is a *degenerate* allocation in so far as there is no uncertainty in the information it expresses. It states simply that the piece of information ϕ holds. Obviously the bottom a.o.p $\nu = \rho_1$ is a deterministic allocations, and so is top a.o.p $\zeta = \rho_0$. Now, for $\phi_1, \phi_2 \in \Psi$ we have

$$\begin{aligned} \rho_{\phi_1} \cdot \rho_{\phi_2}(\psi) &= \bigvee \{ \rho_{\phi_1}(\psi_1) \wedge \rho_{\phi_2}(\psi_2) : \psi \leq \psi_1 \cdot \psi_2 \} \\ &= \left\{ \begin{array}{ll} \top & \text{if } \psi \leq \phi_1 \cdot \phi_2, \\ \perp & \text{otherwise} \end{array} \right\} = \rho_{\phi_1 \cdot \phi_2}(\psi). \end{aligned} \quad (10.7)$$

So, the combination of deterministic allocations of ϕ_1 and ϕ_2 produces the deterministic a.o.p of $\phi_1 \cdot \phi_2$.

Further, for any $\psi \in \Psi$,

$$\epsilon_x(\rho_\phi)(\psi) = \bigvee \{ \rho_\phi(\psi') : \psi \leq \psi' = \epsilon_x(\psi') \}.$$

This equals \top , if there is a $\psi' = \epsilon_x(\psi') \geq \psi$ such that $\psi' \leq \phi$, and \perp otherwise. But, we have $\psi' = \epsilon_x(\psi') \leq \phi$ if and only if $\psi' = \epsilon_x(\psi') \leq \epsilon_x(\phi)$. This shows that $\epsilon_x(\rho_\phi)(\psi) = \rho_{\epsilon_x(\phi)}(\psi)$, hence $\epsilon_x(\rho_\phi) = \rho_{\epsilon_x(\phi)}$. The extraction of a deterministic a.o.p associated with ϕ by x yields the deterministic a.o.p associated with $\epsilon_x(\phi)$.

The mapping $\phi \mapsto \rho_\phi$ is thus an embedding of $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ in $(A_\Psi, D; \leq, \perp, \cdot, \epsilon)$. In this sense, $(A_\Psi, D; \leq, \perp, \cdot, \epsilon)$ extends the information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. By the way, we remark that if $(\Psi, E; \cdot, \circ)$ is an idempotent valuation algebra (Section 5.2), then the corresponding algebra of a.o.p is obviously also an idempotent valuation algebra.

In the next section we pursue the subject by examining the question how random mappings and allocations of probability, and especially their respective information algebras, are related.

10.2 Random Mappings and Allocations

In Section 9.2 it has been shown that a random mapping generates an allocation of probability, which specifies how much belief, according to the information represented by the random mapping, is to be assigned to an element of Φ . In this section the relations between random mappings and allocations of probability will be examined in more detail. In particular, we address the question, whether the operations between random mappings, combination and extraction, are reflected in the corresponding operations of the associated a.o.p, in other words, whether the mapping $\Gamma \mapsto \rho_\Gamma$ is a homomorphism between random mappings and associated allocations of probability.

We start with *simple random variables*. Fix an idempotent generalised information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ and a probability space (Ω, \mathcal{A}, P) . For any simple random variable $\Delta \in \mathcal{R}_s$ defined on this probability space, we have seen that all elements of Ψ and even of I_Ψ have measurable allocations of support $s_\Delta(\psi) \in \mathcal{A}$ and their degree of support is well defined. If we pass in this case from the probability space (Ω, \mathcal{A}, P) to its associated probability algebra (\mathcal{B}, μ) (see Section 9.2), then we can define the *allocation of probability* (a.o.p) associated with the random variable Δ ,

$$\rho_\Delta(\psi) = [s_\Delta(\psi)]$$

for all elements $\psi \in \Psi$ and even for all elements in I_Ψ . Thus, we obtain for the degree of support induced by the random variable Δ ,

$$sp_\Delta(\psi) = P(s_\Delta(\psi)) = \mu(\rho_\Delta(\psi)).$$

Again this holds for all elements of Ψ and even of its ideal completion I_Ψ . The mapping $\rho_\Delta : \Psi \rightarrow \mathcal{B}$ clearly satisfies the properties (A1) and A(2) of an allocation of probability introduced in the previous Section 10.1 (see Theorem 9.1 and (9.7)).

A simple random variable Δ is defined by a partition $\{B_1, \dots, B_n\}$ of Ω consisting of measurable blocks B_i and a mapping defined by $\Delta(\omega) = \psi_i$ for all $\omega \in B_i$ and $i = 1, \dots, n$. We write $\Delta(\omega) = \Delta(B_i)$, if $\omega \in B_i$. To the partition $\{B_1, \dots, B_n\}$ of Ω corresponds a partition $\{[B_1], \dots, [B_n]\}$ of the probability algebra \mathcal{B} . That is, we have $[B_i] \wedge [B_j] = \perp$ if $i \neq j$, and $\vee_{i=1}^n [B_i] = \top$. The simple random variable Δ can also be defined by a mapping $\Delta([B_i]) = \psi_i$ from the partition of \mathcal{B} into Ψ . Its allocation of probability can then also be determined as

$$\rho_\Delta(\psi) = \vee \{[B_i] : \psi \leq \Delta([B_i])\}. \quad (10.8)$$

We note that $\rho_\Delta = \rho_{\Delta \rightarrow}$. So, as far as allocation of probability (and support) is concerned we might as well restrict ourselves to considering *canonical* simple random variables and their information algebra $(\mathcal{R}_{s,c}, D; \leq, \perp, \cdot, \epsilon)$ (see Section 9.1).

We now consider the mapping $\rho : \Delta \mapsto \rho_\Delta$ which maps simple random variables into a.o.p.s. This mapping is a *homomorphism*:

Theorem 10.7 *Let $\Delta_1, \Delta_2, \Delta \in \mathcal{R}_s$ be simple random variables, defined on partitions in a probability algebra (\mathcal{B}, μ) with values in an idempotent generalised information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. Then, for all $\psi \in \Psi$ and $x \in D$,*

$$\rho_{\Delta_1 \cdot \Delta_2}(\psi) = (\rho_{\Delta_1} \cdot \rho_{\Delta_2})(\psi) \quad (10.9)$$

$$\rho_{\epsilon_x(\Delta)}(\psi) = \epsilon_x(\rho_\Delta)(\psi). \quad (10.10)$$

It is understood that in this theorem the combination on the left is the one in the algebra of simple random variables, whereas on the right it is the one in the algebra of a.o.ps. Similarly, the extraction operator ϵ_x on the left is the one in the information algebra $(\mathcal{R}_s, D; \leq, \perp, \cdot, \epsilon)$ of simple random variables, the one on the right is the one in the information algebra $(A_\Psi, D; \leq, \perp, \cdot, \epsilon)$ of a.o.p

Proof. (1) Assume that Δ_1 is defined on the partition $\{B_{1,1}, \dots, B_{1,n}\}$ and Δ_2 on the partition $\{B_{2,1}, \dots, B_{2,m}\}$ of \mathcal{B} . From the definition of an allocation of probability, and the distributive and associative laws for Boolean algebras, we obtain

$$\begin{aligned}
 & (\rho_{\Delta_1} \cdot \rho_{\Delta_2})(\psi) \\
 &= \vee \{ \rho_{\Delta_1}(\psi_1) \wedge \rho_{\Delta_2}(\psi_2) : \psi \leq \psi_1 \cdot \psi_2 \} \\
 &= \vee \{ (\vee \{ B_{1,i} : \psi_1 \leq \Delta_1(B_{1,i}) \}) \\
 &\quad \wedge (\vee \{ B_{2,j} : \psi_2 \leq \Delta_2(B_{2,j}) \}) : \psi \leq \psi_1 \cdot \psi_2 \} \\
 &= \vee \{ \vee \{ B_{1,i} \wedge B_{2,j} \neq \perp : \psi_1 \leq \Delta_1(B_{1,i}), \psi_2 \leq \Delta_2(B_{2,j}) \} : \psi \leq \psi_1 \cdot \psi_2 \} \\
 &= \vee \{ B_{1,i} \wedge B_{2,j} \neq \perp : \psi_1 \leq \Delta_1(B_{1,i}), \psi_2 \leq \Delta_2(B_{2,j}), \psi \leq \psi_1 \cdot \psi_2 \}.
 \end{aligned}$$

But $\psi \leq \psi_1 \cdot \psi_2$, $\psi_1 \leq \Delta_1(B_{1,i})$ and $\psi_2 \leq \Delta_2(B_{2,j})$ if and only if $\psi \leq \Delta_1(B_{1,i}) \cdot \Delta_2(B_{2,j})$. So we conclude that

$$\begin{aligned}
 & (\rho_{\Delta_1} \cdot \rho_{\Delta_2})(\psi) \\
 &= \vee \{ B_{1,i} \wedge B_{2,j} \neq \perp : \psi \leq \Delta_1(B_{1,i}) \cdot \Delta_2(B_{2,j}) \} \\
 &= \vee \{ B_{1,i} \wedge B_{2,j} \neq \perp : \psi \leq (\Delta_1 \cdot \Delta_2)(B_{1,i} \wedge B_{2,j}) \} \quad (10.11) \\
 &= \rho_{\Delta_1 \cdot \Delta_2}(\psi).
 \end{aligned}$$

(2) Assume that Δ is defined on the partition B_1, \dots, B_n of \mathcal{B} . Then $\epsilon_x(\Delta)$ is also defined on B_1, \dots, B_n . The associative law of complete Boolean algebra gives us then,

$$\begin{aligned}
 & \epsilon_x(\rho_\Delta)(\psi) \\
 &= \vee \{ \rho_\Delta(\phi) : \psi \leq \phi = \epsilon_x(\phi) \} \\
 &= \vee \{ \vee \{ B_i : \phi \leq \Delta(B_i) \} : \psi \leq \phi = \epsilon_x(\phi) \} \\
 &= \vee \{ B_i : \psi \leq \phi = \epsilon_x(\phi) \leq \Delta(B_i) \}.
 \end{aligned}$$

But, $\psi \leq \phi = \epsilon_x(\phi) \leq \Delta(B_i)$ holds if and only if $\psi \leq \epsilon_x(\Delta(B_i)) = \epsilon_x(\Delta)(B_i)$. Hence we see that

$$\epsilon_x(\rho_\Delta)(\psi) = \vee \{ B_i : \psi \leq \epsilon_x(\Delta)(B_i) \} = \rho_{\epsilon_x(\Delta)}(\psi).$$

This completes the proof. \square

As far as allocations of probability induced by simple random variables are concerned, this theorem shows that the combination and focusing of

allocations reflects correctly the corresponding operations of the underlying random variables. Let A_s be the image of \mathcal{R}_s , under the mapping ρ . That is A_s is the set of all allocations of probability which are induced by simple random variables in (\mathcal{B}, μ) . The mapping satisfies

$$\begin{aligned}\rho_{\Delta_1 \cdot \Delta_2} &= \rho_{\Delta_1} \cdot \rho_{\Delta_2}, \\ \rho_{\epsilon_x(\Delta)} &= \epsilon_x(\rho_\Delta).\end{aligned}\tag{10.12}$$

Also the vacuous random variable U maps to the vacuous allocation ν and the null random variable N to ζ . Thus we conclude that the map $\Delta \mapsto \rho_\Delta$ is a homomorphism between $(\mathcal{R}_s, D; \leq, \perp, \cdot, \epsilon)$ and $(A_\Psi, D; \leq, \perp, \cdot, \epsilon)$ and that $(A_s, D; \leq, \perp, \cdot, \epsilon)$ is a subalgebra of the information algebra $(A_\Psi, D; \leq, \perp, \cdot, \epsilon)$. We remark that if we restrict the mapping ρ to *canonical* random variables, then the mapping $\Delta \mapsto \rho_\Delta$ becomes an *embedding*.

Now we turn to *generalised random variables* Γ . We remind that they can be identified with certain random mappings into the ideal completion $(I_\Psi, D; \leq, \perp, \cdot, \epsilon)$ of the information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ (see Section 9.3) and as such their allocation of probability is defined by $\rho_\Gamma(\psi) = \rho_0(s_\Gamma(\psi))$ or $\rho_\Gamma = \rho_0 \circ s_\Gamma$ (see Section 9.2). We remind that this covers also the important case of *algebraic* information algebras $(\Psi, D; \leq, \perp, \cdot, \epsilon)$, where the simple random variables have finite values in Ψ_f , if $(\Psi_f, D; \leq, \perp, \cdot, \epsilon)$ is a subalgebra of $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. Now we show that the a.o.p of a generalised random variable can also be obtained as the limit of the a.o.p of the simple random variables it dominates.

Theorem 10.8 *For all generalised random variables Γ ,*

$$\rho_\Gamma = \bigvee \{\rho_\Delta : \Delta \leq \Gamma\}.\tag{10.13}$$

Proof. Fix an element $\psi \in \Psi$ and consider a measurable subset $A \subseteq s_\Gamma(\psi)$. We define a simple random variable

$$\Delta(\omega) = \begin{cases} \psi & \text{if } \omega \in A, \\ 1 & \text{otherwise.} \end{cases}$$

Then certainly $\Delta(\omega) \leq \Gamma(\omega)$ for all $\omega \in \Omega$, hence $\Delta \leq \Gamma$. Furthermore we have $\rho_\Delta(\psi) = [A]$. This implies that

$$\bigvee \{\rho_\Delta(\psi) : \Delta \leq \Gamma\} \geq \bigvee \{[A] : A \subseteq s_\Gamma(\psi), A \in \mathcal{A}\} = \rho_0(s_\Gamma(\psi)).$$

Conversely, for all $\Delta \leq \Gamma$ it holds that $s_\Delta(\psi) \subseteq s_\Gamma(\psi)$ and that $s_\Delta(\psi) \in \mathcal{A}$. Therefore, we conclude that

$$\bigvee \{\rho_\Delta(\psi) : \Delta \leq \Gamma\} \leq \bigvee \{[A] : A \subseteq s_\Gamma(\psi), A \in \mathcal{A}\} = \rho_0(s_\Gamma(\psi)).$$

This proves that $\rho_\Gamma(\psi) = \bigvee \{\rho_\Delta(\psi) : \Delta \leq \Gamma\}$ for all $\psi \in \Psi$, hence (10.13) holds. \square

Theorem 10.8 shows that the a.o.p of a generalised random variable is in the ideal completion of the information algebra $(A_s, D; \leq, \perp, \cdot, \epsilon)$ of simple a.o.p. This ideal completion contains *allocations of probability* $\rho_\Gamma : \mathcal{B} \rightarrow I_\Psi$ of the random mappings associated with generalised random variables. The ideal completion of $(A_s, D; \leq, \perp, \cdot, \epsilon)$ is an algebraic information algebra and (10.13) shows that the mapping $\Gamma \mapsto \rho_\Gamma$ is *continuous*. It is in fact a homomorphism between the algebra of generalised random variables and their a.o.p as the following theorem shows:

Theorem 10.9 *Let $\Gamma, \Gamma_1, \Gamma_2$ be generalised random variables and $x \in D$. Then*

$$\begin{aligned}\rho_{\Gamma_1 \cdot \Gamma_2} &= \rho_{\Gamma_1} \cdot \rho_{\Gamma_2}, \\ \rho_{\epsilon_x(\Gamma)} &= \epsilon_x(\rho_\Gamma).\end{aligned}$$

The operations on the left hand side of these identities belong to the algebra of generalised random variables, whereas those on the right hand side to the algebra of a.o.p.

Proof. We have to show that

$$\begin{aligned}\rho_{\Gamma_1 \cdot \Gamma_2}(\psi) &= (\rho_{\Gamma_1} \cdot \rho_{\Gamma_2})(\psi), \\ \rho_{\epsilon_x(\Gamma)}(\psi) &= \epsilon_x(\rho_\Gamma)(\psi),\end{aligned}$$

for all $\psi \in \Phi$.

(1) We noted above that the mapping $\Gamma \mapsto \rho_\Gamma$ is continuous. Therefore, using (9.20) and continuity, Δ denoting always simple random variables, we have

$$\rho_{\Gamma_1 \cdot \Gamma_2} = \rho_{\bigvee \{\Delta_1 \cdot \Delta_2 : \Delta_1 \leq \Gamma_1, \Delta_2 \leq \Gamma_2\}} = \bigvee \{\rho_{\Delta_1 \cdot \Delta_2} : \Delta_1 \leq \Gamma_1, \Delta_2 \leq \Gamma_2\}.$$

On the other hand, for every $\psi \in \Psi$, we obtain, using Theorems 10.8 and 10.8, and the associative and distributive laws of Boolean algebras,

$$\begin{aligned}(\rho_{\Gamma_1} \cdot \rho_{\Gamma_2})(\psi) &= \bigvee \{\rho_{\Gamma_1}(\psi_1) \wedge \rho_{\Gamma_2}(\psi_2) : \psi \leq \psi_1 \cdot \psi_2\} \\ &= \bigvee \{(\bigvee \{\rho_{\Delta_1}(\psi_1) : \Delta_1 \leq \Gamma_1\}) \\ &\quad \wedge (\bigvee \{\rho_{\Delta_2}(\psi_2) : \Delta_2 \leq \Gamma_2\}) : \psi \leq \psi_1 \cdot \psi_2\} \\ &= \bigvee \{\rho_{\Delta_1}(\psi_1) \wedge \rho_{\Delta_2}(\psi_2) : \Delta_1 \leq \Gamma_1, \Delta_2 \leq \Gamma_2, \psi \leq \psi_1 \cdot \psi_2\} \\ &= \bigvee \{\bigvee \{\rho_{\Delta_1}(\psi_1) \wedge \rho_{\Delta_2}(\psi_2) : \psi \leq \psi_1 \cdot \psi_2\} : \Delta_1 \leq \Gamma_1, \Delta_2 \leq \Gamma_2\} \\ &= \bigvee \{(\rho_{\Delta_1} \cdot \rho_{\Delta_2})(\psi) : \Delta_1 \leq \Gamma_1, \Delta_2 \leq \Gamma_2\} \\ &= \bigvee \{\rho_{\Delta_1 \cdot \Delta_2}(\psi) : \Delta_1 \leq \Gamma_1, \Delta_2 \leq \Gamma_2\}.\end{aligned}$$

This proves that $\rho_{\Gamma_1 \cdot \Gamma_2} = \rho_{\Gamma_1} \cdot \rho_{\Gamma_2}$.

(2) Again by continuity, we obtain from (9.21)

$$\rho_{\epsilon_x(\Gamma)} = \rho_{\bigvee\{\epsilon_x(\Delta) : \Delta \leq \Gamma\}} = \bigvee\{\rho_{\epsilon_x(\Delta)} : \Delta \leq \Gamma\}.$$

But, we have also, by Theorem 10.7 (10.10) and Theorem 10.8 ,

$$\begin{aligned} \epsilon_x(\rho_\Gamma)(\phi) &= \bigvee\{\rho_\Gamma(\psi) : \phi \leq \psi = \epsilon_x(\psi)\} \\ &= \bigvee\{\bigvee\{\rho_\Delta(\psi) : \Delta \leq \Gamma\} : \phi \leq \psi = \epsilon_x(\psi)\} \\ &= \bigvee\{\bigvee\{\rho_\Delta(\psi) : \phi \leq \psi = \epsilon_x(\psi)\} : \Delta \leq \Gamma\} \\ &= \bigvee\{\rho_{\epsilon_x(\Delta)}(\phi) : \Delta \leq \Gamma\}. \end{aligned}$$

This proves that $\rho_{x(\Gamma)} = \epsilon_x(\rho_\Gamma)$. \square

The following is a remarkable property of generalised random variables, which we formulate in the framework of algebraic information algebras. The interest of this theorem will become clear later especially in relation to support functions, see Chapter 11.

Theorem 10.10 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be an algebraic information algebra with finite elements Ψ_f and $(\Psi_f, D; \leq, \perp, \cdot, \epsilon)$ a subalgebra of $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. Let Γ be a generalised random variable in $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. Then, for any directed set $X \subseteq \Psi$,*

$$\rho_\Gamma(\bigvee X) = \bigwedge_{\psi \in X} \rho_\Gamma(\psi). \quad (10.14)$$

Proof. We prove first the identity

$$\rho_\Delta(\phi) = \bigwedge\{\rho_\Delta(\psi) : \psi \in \Psi_f, \psi \leq \phi\}. \quad (10.15)$$

for simple random variables Δ . Using the convention introduced at the beginning of the section, we write $\Delta([B_i]) = \psi_i$, where the $[B_i]$ form a partition of \mathcal{B} for $i = 1, \dots, n$. Then its a.o.p is given by $\rho_\Delta(\psi) = \bigvee\{[B_i] : \psi \leq \psi_i\}$ (see (10.8)). Using this, we obtain

$$\bigwedge\{\rho_\Delta(\psi) : \psi \in \Psi_f, \psi \leq \phi\} = \bigwedge\{\bigvee_{\psi \leq \psi_i} [B_i] : \psi \in \Psi_f, \psi \leq \phi\}$$

Since the partition $[B_i]$ of \mathcal{B} is finite, the join on the right hand side extends for every ψ only over a finite number of elements $[B_i]$. Further, as ψ increases, the number of these elements can only decrease. But in $\rho_\Delta(\phi) = \bigvee\{[B_i] : \phi \leq \psi_i\}$ also only a finite number of elements $[B_i]$ appear and this number must be less or equal to the number for any $\psi \leq \phi$. So, as ψ increases towards ϕ , a minimal number of elements must be attained for some $\psi_0 \leq \phi$. Say this number is m and assume that the elements are numbered as $[B_1], \dots, [B_m]$. Then we conclude that the infimum

$\bigwedge\{\rho_\Gamma(\psi) : \psi \in \Psi_f, \psi \leq \phi\}$ equals $\bigvee_{i=1}^m [B_i]$. Now, for all ψ such that $\psi_0 \leq \psi \leq \phi$ we have $\psi \leq \psi_1, \dots, \psi_m$. Since $\phi = \bigvee_{\psi_0 \leq \psi \leq \phi} \psi$, we conclude that $\phi \leq \psi_1, \dots, \psi_m$. But this means that $\rho_\Delta(\phi) = \bigvee_{i=1}^m [B_i]$ and this proves (10.15).

Next, we extend (10.15) to any generalised random variables $\Gamma = \bigvee\{\Delta : \Delta \in \mathcal{R}_s, \Delta \leq \Gamma\}$. For this purpose we use the distributive law in the complete Boolean algebra \mathcal{B} :

$$\begin{aligned}
\rho_\Gamma(\phi) &= \bigvee\{\rho_\Delta(\phi) : \Delta \in \mathcal{R}_s, \Delta \leq \Gamma\} \\
&= \bigvee\{\bigwedge\{\rho_\Delta(\psi) : \psi \in \Psi_f, \psi \leq \phi\} : \Delta \in \mathcal{R}_s, \Delta \leq \Gamma\} \\
&= \bigwedge\{\bigvee\{\rho_\Delta(\psi) : \Delta \in \mathcal{R}_s, \Delta \leq \Gamma\} : \psi \in \Psi_f, \psi \leq \phi\} \\
&= \bigwedge\{\rho_\Gamma(\psi) : \psi \in \Psi_f, \psi \leq \phi\}
\end{aligned} \tag{10.16}$$

To conclude, let $X \subseteq \Psi$ be directed. Consider $\psi \in X$. Then $\psi \leq \bigvee X$, hence $\rho_\Gamma(\psi) \geq \rho_\Gamma(\bigvee X)$, and it follows that $\bigwedge_{\psi \in X} \rho_\Gamma(\psi) \geq \rho_\Gamma(\bigvee X)$. Further, if η is a finite element and $\eta \leq \bigvee X$, then there is a $\psi \in X$ such that $\eta \leq \psi$. This implies that $\rho_\Gamma(\eta) \geq \rho_\Gamma(\psi)$. From this we conclude, using (10.16)

$$\begin{aligned}
\rho_\Gamma(\bigvee X) &= \bigwedge\{\rho_\Gamma(\eta) : \eta \in \Psi_f, \eta \leq \bigvee X\} \\
&\geq \bigwedge_{\psi \in X} \rho_\Gamma(\psi).
\end{aligned}$$

This proves finally (10.14). \square

Following (Shafer, 1979) we call an allocation of probability satisfying (10.14) *condensable*. Thus, the a.o.ps associated with generalised random variables are condensable.

Next we examine the case of *random variables* and their allocations of probability. According to Section 9.3, random variables Γ are ideals in $I_{\mathcal{R}_s}$ and as random mappings $\Gamma(\omega) = \bigvee_{i=1}^\infty \Delta_i(\omega)$, where Δ_i are simple random variables, they map into I_Ψ , or more precisely into $\sigma(\Psi) \subseteq I_\Psi$. This is equivalent to looking at an *algebraic* information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ and considering random variables on the finite elements Ψ_f . By the Representation Theorem 7.5 the information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is isomorphic to the ideal completion $(I_{\Psi_f}, D; \leq, \perp, \cdot, \epsilon)$ of the subalgebra of the finite elements $(\Psi_f, D; \leq, \perp, \cdot, \epsilon)$. In the sequel, we consider this case.

A random variable Γ is then the join (or the limit) of a monotone non-decreasing sequence of simple random variables Δ_i with $\Delta_1 \leq \Delta_2 \leq \dots$, $\Gamma = \bigvee_{i=1}^\infty \Delta_i$. The simple random variables take values in Ψ_f , and the

random variable Γ in Ψ . By Lemma 9.6 a random variable Γ is also a generalised random variable. Therefore Theorem 10.9 applies also to random variables. So, the mapping $\Gamma \mapsto \rho_\Gamma$ is a *homomorphism* of the information algebra $(\mathcal{R}_\sigma, D; \leq, \perp, \cdot, \epsilon)$ of random variables into the information algebra $(A_\Psi, D; \leq, \perp, \cdot, \epsilon)$ of a.o.ps.

We are going to show more, namely that the map $\Gamma \mapsto \rho_\Gamma$ is a σ -homomorphism from the σ -information algebra $(\mathcal{R}_\sigma, D; \leq, \perp, \cdot, \epsilon)$ into the information algebra $(A_\Psi, D; \leq, \perp, \cdot, \epsilon)$.

Theorem 10.11 *Suppose $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ to be an idempotent generalised information algebra, and $\Gamma_i \in \mathcal{R}_\sigma$ for $i = 1, 2, \dots$. Then*

$$\rho_{\bigvee_{i=1}^{\infty} \Gamma_i} = \bigvee_{i=1}^{\infty} \rho_{\Gamma_i}. \quad (10.17)$$

Proof. Since the mapping $\Gamma \mapsto \rho_\Gamma$ is a homomorphism, it maintains order. As a random variable, Γ equals $\bigvee_{i=1}^{\infty} \Delta_i$, where Δ_i form a monotone sequence of simple random variables. Since Γ is also a generalised random variable, we have by (10.13) $\rho_\Gamma = \bigvee \{\rho_\Delta : \rho_\Delta \in \mathcal{R}_s, \Delta \leq \Gamma\}$. The monotone sequence Δ_i is directed in \mathcal{R} . By compactness there is for every $\Delta \leq \Gamma$ an index j so that $\Delta \leq \Delta_j$. This implies $\rho_\Delta \leq \rho_{\Delta_j}$ from which it follows that $\rho_\Gamma \leq \bigvee_{i=1}^{\infty} \rho_{\Delta_i}$. The converse inequality is evident. So we conclude that

$$\rho_\Gamma = \bigvee_{i=1}^{\infty} \rho_{\Delta_i} \quad (10.18)$$

if $\Gamma = \bigvee_{i=1}^{\infty} \Delta_i$.

Consider now the generalised random variables Γ_i for $i = 1, 2, \dots$ and $\Gamma = \bigvee_{i=1}^{\infty} \Gamma_i$. Let $\Gamma_i = \bigvee_{j=1}^{\infty} \Delta_{i,j}$, where for every $i = 1, 2, \dots$ the sequence $\Delta_{i,1}, \Delta_{i,2}, \dots$ is a monotone sequence of simple random variables. Then

$$\Gamma = \bigvee_{i=1}^{\infty} \bigvee_{j=1}^{\infty} \Delta_{i,j}.$$

In the standard way, we define $\Delta_i = \bigvee_{h=1}^i \bigvee_{j=1}^h \Delta_{h,j}$. The Δ_i form a monotone sequence of simple random variables and $\Gamma = \bigvee_{i=1}^{\infty} \Delta_i$. By (10.18), and the homomorphism between simple random variables and their a.o.ps we obtain

$$\begin{aligned} \rho_\Gamma &= \bigvee_{i=1}^{\infty} \rho_{\Delta_i} = \bigvee_{i=1}^{\infty} \left(\bigvee_{h=1}^i \bigvee_{j=1}^h \rho_{\Delta_{h,j}} \right) \\ &= \bigvee_{i=1}^{\infty} \left(\bigvee_{j=1}^{\infty} \rho_{\Delta_{i,j}} \right) = \bigvee_{i=1}^{\infty} \rho_{\Gamma_i}. \end{aligned}$$

This proves (10.17). \square

As a preparation for an interpretation of this result, we remark that for a σ -information algebra the following general result holds:

Lemma 10.1 *Suppose $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ to be a σ -information algebra and Γ a random mapping. Then*

$$s_\Gamma\left(\bigvee_{i=1}^{\infty} \psi_i\right) = \bigcap_{i=1}^{\infty} s_\Gamma(\psi_i). \quad (10.19)$$

Proof. We have

$$s_\Gamma\left(\bigvee_{i=1}^{\infty} \psi_i\right) = \{\omega \in \Omega : \bigvee_{i=1}^{\infty} \psi_i \leq \Gamma(\omega)\}.$$

Let $\psi = \bigvee_{i=1}^{\infty} \psi_i$. Since $\psi_i \leq \psi$ we conclude that $s_\Gamma(\psi) \subseteq s_\Gamma(\psi_i)$, hence $s_\Gamma(\psi) \subseteq \bigcap_{i=1}^{\infty} s_\Gamma(\psi_i)$. On the other hand, consider $\omega \in \bigcap_{i=1}^{\infty} s_\Gamma(\psi_i)$, that is $\psi_i \leq \Gamma(\omega)$ for all $i = 1, 2, \dots$. Then we have $\bigvee_{i=1}^{\infty} \psi_i = \psi \leq \Gamma(\omega)$, hence $\omega \in s_\Gamma(\psi)$. This shows that $s_\Gamma(\psi) \supseteq \bigcap_{i=1}^{\infty} s_\Gamma(\psi_i)$ and this proves (10.19). \square

Since for any random variable Γ and every $\psi \in \Psi$, we have $\rho_\Gamma(\psi) = \rho_0(s_\Gamma(\psi))$ and the mapping ρ_0 is a σ -homomorphism from the power set of Ω onto \mathcal{B} (see Theorem 9.2) it follows also from (10.19)

$$\rho_\Gamma\left(\bigvee_{i=1}^{\infty} \psi_i\right) = \bigwedge_{i=1}^{\infty} \rho_\Gamma(\psi_i).$$

An allocation of probability, which satisfies this identity is called a σ -allocation of probability. Thus, a random variables induces a σ -a.o.p. Let A_σ denote the image of \mathcal{R}_σ under the mapping $\Gamma \mapsto \rho_\Gamma$ in A_Ψ .

Next we show that continuity of extraction is also satisfied in the algebra $(A_\sigma, D; \leq, \perp, \cdot, \epsilon)$:

Theorem 10.12 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be an algebraic information algebra, and $\Gamma_i \in \mathcal{R}_\sigma$ for $i = 1, 2, \dots$ a monotone sequence of random variables, $\Gamma_1 \leq \Gamma_2 \leq \dots$. Then for every $x \in D$,*

$$\epsilon_x\left(\bigvee_{i=1}^{\infty} \rho_{\Gamma_i}\right) = \bigvee_{i=1}^{\infty} \epsilon_x(\rho_{\Gamma_i}). \quad (10.20)$$

Proof. The proof is based on the continuity of extraction in the σ -information algebra $(\mathcal{R}_\sigma, D; \leq, \perp, \cdot, \epsilon)$ of random variables, see Theorem 9.5,

$$\epsilon_x\left(\bigvee_{i=1}^{\infty} \Gamma_i\right) = \bigvee_{i=1}^{\infty} \epsilon_x(\Gamma_i).$$

Take the a.o.p of both sides. Using the fact that the mapping is a homomorphism of generalised random variables, Theorem 10.9, and Theorem 10.11, this leads on the left hand to

$$\rho_{\epsilon_x(\bigvee_{i=1}^{\infty} \Gamma_i)} = \epsilon_x(\rho_{\bigvee_{i=1}^{\infty} \Gamma_i}) = \epsilon_x(\bigvee_{i=1}^{\infty} \rho_{\Gamma_i}).$$

On the right hand side we obtain by the same argument

$$\rho_{\bigvee_{i=1}^{\infty} \epsilon_x(\Gamma_i)} = \bigvee_{i=1}^{\infty} \rho_{\epsilon_x(\Gamma_i)} = \bigvee_{i=1}^{\infty} \epsilon_x(\rho_{\Gamma_i})$$

This proves the identity (10.20). \square

What can be said about the mapping $\Gamma \mapsto \rho_{\Gamma}$ for random mappings Γ in general? Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be an information algebra, (Ω, \mathcal{A}, P) a probability space and $\Gamma : \Omega \rightarrow \Psi$ a random mapping. The mapping $\Gamma \mapsto \rho_{\Gamma}$ is obviously *order-preserving*: $\Gamma_1 \leq \Gamma_2$ means that $\Gamma_1(\omega) \leq \Gamma_2(\omega)$ for all $\omega \in \Omega$. This implies that $s_{\Gamma_1}(\psi) \subseteq s_{\Gamma_2}(\psi)$ for all $\psi \in \Psi$, and from this it follows that $\rho_{\Gamma_1}(\psi) = \rho_0(s_{\Gamma_1}(\psi)) \leq \rho_0(s_{\Gamma_2}(\psi)) = \rho_{\Gamma_2}(\psi)$ for all $\psi \in \Psi$, hence $\rho_{\Gamma_1} \leq \rho_{\Gamma_2}$.

But the mapping is no more a homomorphism. In fact, let Γ_1 and Γ_2 be two random mappings. Then the support of the combination of these random mappings is

$$\begin{aligned} s_{\Gamma_1 \cdot \Gamma_2}(\psi) &= \{\omega \in \Omega : \psi \leq \Gamma_1(\omega) \cdot \Gamma_2(\omega)\} \\ &= \bigcup \{\omega : \psi_1 \leq \Gamma_1(\omega), \psi_2 \leq \Gamma_2(\omega), \psi \leq \psi_1 \cdot \psi_2\} \\ &= \bigcup \{s_{\Gamma_1}(\psi_1) \cap s_{\Gamma_2}(\psi_2) : \psi \leq \psi_1 \cdot \psi_2\}. \end{aligned}$$

Note that for any index set I , $H_i \subseteq \bigcup_{i \in I} H_i$, hence $\rho_0(H_i) \leq \rho_0(\bigcup_{i \in I} H_i)$ and therefore $\bigvee_{i \in I} \rho_0(H_i) \leq \rho_0(\bigcup_{i \in I} H_i)$. This implies then for all $\psi \in \Psi$

$$\begin{aligned} \rho_{\Gamma_1 \cdot \Gamma_2}(\psi) &= \rho_0(\bigcup \{s_{\Gamma_1}(\psi_1) \cap s_{\Gamma_2}(\psi_2) : \psi \leq \psi_1 \cdot \psi_2\}) \\ &\geq \bigvee \{\rho_0(s_{\Gamma_1}(\psi_1) \cap s_{\Gamma_2}(\psi_2)) : \psi \leq \psi_1 \cdot \psi_2\} \\ &= \bigvee \{\rho_0(s_{\Gamma_1}(\psi_1)) \wedge \rho_0(s_{\Gamma_2}(\psi_2)) : \psi \leq \psi_1 \cdot \psi_2\} \\ &= \bigvee \{\rho_{\Gamma_1}(\psi_1) \wedge \rho_{\Gamma_2}(\psi_2) : \psi \leq \psi_1 \cdot \psi_2\} \\ &= (\rho_{\Gamma_1} \cdot \rho_{\Gamma_2})(\psi). \end{aligned} \tag{10.21}$$

So, we have $\rho_{\Gamma_1 \cdot \Gamma_2} \geq \rho_{\Gamma_1} \cdot \rho_{\Gamma_2}$. Equality holds only in particular cases, like for instance for generalised or σ -random variables. Since $\rho_{\Gamma_1 \cdot \Gamma_2}$ allocates more probability to a hypothesis $\psi \in \Psi$ than $\rho_{\Gamma_1} \cdot \rho_{\Gamma_2}$ does, it seems that by the map to the allocation of probability some information is lost in general.

Consider also extraction, that is a random mapping Γ and $x \in D$. Then, since $(\epsilon_x(\Gamma))(\omega) = \epsilon_x(\Gamma(\omega))$,

$$\begin{aligned} s_{\epsilon_x(\Gamma)}(\psi) &= \{\omega \in \Omega : \psi \leq \epsilon_x(\Gamma(\omega))\} \\ &= \bigcup \{s_\Gamma(\phi) : \phi = \epsilon_x(\phi), \psi \leq \phi\}. \end{aligned} \tag{10.22}$$

Thus, we obtain for the a.o.p of $\epsilon_x(\Gamma)$,

$$\begin{aligned} \rho_{\epsilon_x(\Gamma)}(\psi) &= \rho_0\left(\bigcup \{s_\Gamma(\phi) : \psi \leq \phi = \epsilon_x(\phi)\}\right) \\ &\geq \bigvee \{\rho_0(s_\Gamma(\phi)) : \psi \leq \phi = \epsilon_x(\phi)\} \\ &= \bigvee \{\rho_\Gamma(\phi) : \psi \leq \phi = \epsilon_x(\psi)\} \\ &= (\epsilon_x(\rho_\Gamma))(\psi). \end{aligned}$$

So, here we find that $\rho_{\epsilon_x(\Gamma)} \geq \epsilon_x(\rho_\Gamma)$ and again equality holds only in particular cases. This is a second indication that the random mapping Γ contains more information than its a.o.p ρ_Γ .

Chapter 11

Support Functions

11.1 Characterisation

As we have noted in Chapter 9, we may consider a random mapping Γ as information, that is, $\Gamma(\omega)$ is a “piece of information”, which can be asserted, provided ω is the sample element chosen by a chance process, or the “correct” assumption in a set of possible assumptions Ω . Here, information $\Gamma(\omega)$ may either be an element of the set Ψ of an idempotent generalised information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ or else an *ideal* of Ψ , hence an element of the ideal completion $(I_\Psi, D; \leq, \perp, \cdot, \epsilon)$ of $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. We have defined the *allocation of support* $s_\Gamma(\psi)$ of a random mapping as the set of elements $\omega \in \Omega$, which imply ψ , i.e. such that ψ belongs to the ideal $\Gamma(\omega)$, $\psi \in \Gamma(\omega)$ or $\psi \leq \Gamma(\omega)$, see Sections 9.1 and 9.2. Any $\omega \in s_\Gamma(\psi)$ is an assumption, i.e. an argument, which permits to infer the piece of information ψ in the light of the random mapping Γ . So, the larger the set $s_\Gamma(\psi)$, the more arguments are available to support ψ . Or, more to the point, the more probable, the more likely it is that the correct, but unknown assumption ω belongs to $s_\Gamma(\psi)$, the stronger the hypothesis ψ is supported. This probability was denoted by $sp_\Gamma(\psi)$ and called the *degree of support* of a hypothesis allocated by a random mapping Γ . We refer to Chapter 9 for this point of view. The degrees of support can be seen as a numerical map or function $sp_\Gamma : \Psi \rightarrow [0, 1]$ of Ψ into the unit interval. The goal of this chapter is to study this function.

We do not exclude in this chapter that $\Gamma(\omega) = 0$ for some ω . This represents improper information, which can be interpreted as contradictory information. Under semantic aspects such improper information could and should be excluded. We refer to Section 9.1 for a discussion of this issue in the context of simple random functions. But for the present discussion this is not essential. If $\Gamma(\omega) \neq 0$ for all ω , the random mapping is called *normalised*.

Consider then a random mapping $\Gamma : \Omega \rightarrow \Psi$ from a probability space (Ω, \mathcal{A}, P) into an idempotent generalised information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$.

The corresponding support is defined for any $\psi \in \Psi$ as

$$s_\Gamma(\psi) = \{\omega \in \Omega : \psi \leq \Gamma(\omega)\}.$$

The set s_Γ thus contains all assumptions ω for which $\Gamma(\omega)$ implies ψ . The following theorem collects a few elementary properties of the mapping $s_\Gamma : \Psi \rightarrow \mathcal{P}(\Omega)$ (see also Theorem 9.1):

Theorem 11.1 *If $\Gamma : \Omega \rightarrow \Psi$, then*

1. $s_\Gamma(1) = \Omega$,
2. *If $\phi \leq \psi$, then $s_\Gamma(\psi) \subseteq s_\Gamma(\phi)$,*
3. $s_\Gamma(\phi \cdot \psi) = s_\Gamma(\phi) \cap s_\Gamma(\psi)$ *for all $\psi, \phi \in \Psi$,*
4. *if Γ is normalised, then $s_\Gamma(0) = \emptyset$.*

Proof. (1) follows since 1 is the least element in Ψ , hence $1 \leq \Gamma(\omega)$ for all $\omega \in \Omega$. (2) is obvious. (3) follows, since $\phi, \psi \leq \Gamma(\omega)$ if and only if $\phi \cdot \psi \leq \Gamma(\omega)$ and (4) follows from the definition of a normalised random mapping. \square

Sometimes Ψ may be a σ -semilattice or even a complete lattice, for instance, if $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is an algebraic or continuous information algebra. Then something more can be said about the support of a random mapping.

Theorem 11.2 *Let $\Gamma : \Omega \rightarrow \Psi$ be a random mapping.*

1. *If Ψ is a σ -semilattice, $\psi_1, \psi_2, \dots \in \Psi$, then*

$$s_\Gamma\left(\bigvee_{i=1}^{\infty} \psi_i\right) = \bigcap_{i=1}^{\infty} s_\Gamma(\psi_i). \quad (11.1)$$

2. *If Ψ is a complete lattice, $X \subseteq \Psi$, then*

$$s_\Gamma\left(\bigvee X\right) = \bigcap_{\psi \in X} s_\Gamma(\psi). \quad (11.2)$$

Proof. (1) We have $\psi_1, \psi_2, \dots \leq \Gamma(\omega)$ if and only if $\bigvee_{i=1}^{\infty} \psi_i \leq \Gamma(\omega)$. This implies (11.1).

(2) Similarly, we have $\psi \leq \Gamma(\omega)$ for all $\psi \in X$ if and only if $\bigvee X \leq \Gamma(\omega)$ and this implies (11.2). \square

We want to make use of the probability space (Ω, \mathcal{A}, P) to judge the likelihood that a random mapping Γ supports a hypothesis $\psi \in \Psi$. The degree of support $sp_\Gamma(\psi)$ of an element $\psi \in \Psi$ is measured by the probability of its support $s_\Gamma(\psi)$, provided this probability is defined. But this is the case only if $s_\Gamma(\psi) \in \mathcal{A}$. Therefore, we define:

Definition 11.1 If $\Gamma : \Omega \rightarrow \Psi$ is a random mapping from a probability space (Ω, \mathcal{A}, P) into an information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$, then $\psi \in \Psi$ is called Γ -measurable, if $s_\Gamma(\psi) \in \mathcal{A}$.

The set of all Γ -measurable elements $\psi \in \Psi$ will be denoted by \mathcal{E}_Γ .

Theorem 11.3 For any random mapping Γ , \mathcal{E}_Γ is a subsemilattice of Ψ , containing 1; if Γ is normalised, then 0 belongs to \mathcal{E}_Γ too. Further, if Ψ is a σ -semilattice, then \mathcal{E}_Γ is a σ -semilattice.

Proof. The first part of the theorem follows from the definition of \mathcal{E}_Γ and Theorem 11.1. The second part follows from Theorem 11.2. \square

On the semilattice \mathcal{E}_Γ we define $sp_\Gamma(\psi) = P(s_\Gamma(\psi))$. Thus, sp_Γ is a function with values in $[0, 1]$, defined on \mathcal{E}_Γ . This function is called the *support function* of the random mapping Γ . The next theorem collects the basic properties of this function.

Theorem 11.4 Let Γ be a random mapping from the probability space (Ω, \mathcal{A}, P) into the idempotent generalised information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$, and sp_Γ the associated support function, defined on \mathcal{E}_Γ . Then sp_Γ has the following properties:

1. $sp_\Gamma(1) = 1$.
2. If $\psi_1, \dots, \psi_m \geq \psi$, $\psi_1, \dots, \psi_m, \psi \in \mathcal{E}_\Gamma$,

$$sp_\Gamma(\psi) \geq \sum_{\emptyset \neq I \subseteq \{1, \dots, m\}} (-1)^{|I|+1} sp_\Gamma(\vee_{i \in I} \psi_i). \quad (11.3)$$

3. If \mathcal{E}_Γ is a σ -semilattice, and if $\psi_1 \leq \psi_2 \leq \dots \in \mathcal{E}_\Gamma$, then

$$sp_\Gamma\left(\bigvee_{i=1}^{\infty} \psi_i\right) = \lim_{i \rightarrow \infty} sp_\Gamma(\psi_i). \quad (11.4)$$

4. If Γ is normalised, then $sp_\Gamma(0) = 0$.

Proof. (1) and (4) follow from Theorem 11.1 (1) and (4).

(2) Note that by Theorem 11.1 (2) we have $sp_\Gamma(\vee_{i \in I} \psi_i) = P(s_\Gamma(\vee_{i \in I} \psi_i)) = P(\cap_{i \in I} s_\Gamma(\psi_i))$ for a finite index set I . On the right hand side of (11.3) we have then by the inclusion-exclusion formula of probability theory,

$$\sum_{\emptyset \neq I \subseteq \{1, \dots, m\}} (-1)^{|I|+1} P(\cap_{i \in I} s_\Gamma(\psi_i)) = P(\cup_{i=1}^m s_\Gamma(\psi_i)).$$

But $\psi \leq \psi_1, \dots, \psi_m$ implies $s_\Gamma(\psi) \supseteq s_\Gamma(\psi_i)$, hence

$$s_\Gamma(\psi) \supseteq \cup_{i=1}^m s_\Gamma(\psi_i)$$

This implies (11.3)

(3) In this case $\bigvee_{i=1}^{\infty} \psi_i \in \mathcal{E}_{\Gamma}$. Further, by Theorem 11.2, $sp_{\Gamma}(\bigvee_{i=1}^{\infty} \psi_i) = P(s_{\Gamma}(\bigvee_{i=1}^{\infty} \psi_i)) = P(\bigcap_{i=1}^{\infty} s_{\Gamma}(\psi_i))$. Now, $\psi_1 \leq \psi_2 \leq \dots$ implies $s_{\Gamma}(\psi_1) \supseteq s_{\Gamma}(\psi_2) \supseteq \dots$ (Theorem 11.1 (2)). By the continuity of probability it follows that $P(\bigcap_{i=1}^{\infty} s_{\Gamma}(\psi_i)) = \lim_{i \rightarrow \infty} P(s_{\Gamma}(\psi_i))$. This proves (11.4). \square

As a consequence we deduce from (2) of the theorem above that for $\phi \leq \psi$ we have $sp_{\Gamma}(\psi) \leq sp_{\Gamma}(\phi)$. Thus the function sp_{Γ} is monotone. In fact a function satisfying property (2) of the theorem above is called *monotone of order ∞* (Choquet, 1953–1954; Choquet, 1969).

In Section 9.2 we proposed to extend the support function of a random mapping Γ beyond the measurable elements by $sp_{\Gamma}(\psi) = \mu(\rho_{\Gamma}(\psi))$, where $\rho_{\Gamma}(\psi) = \rho_0(s_{\Gamma}(\psi))$ is the allocation of probability associated with the random mapping Γ and (μ, \mathcal{B}) is the probability algebra associated with the probability space (Ω, \mathcal{A}, P) . Now, any allocation of probability $\rho : \mathcal{B} \rightarrow \Psi$ generates a function $sp = \mu \circ \rho$ which satisfies properties (1) and (2) of Theorem 11.4 as stated in Theorem 11.5 below. Therefore, in particular the function $sp_{\Gamma} = \mu \circ \rho_{\Gamma}$, which is defined on Ψ , and even I_{Ψ} has the properties stated in Theorem 11.4.

Theorem 11.5 *Let (μ, \mathcal{B}) be a probability algebra, $\rho : \Psi \rightarrow \mathcal{B}$ an allocation of probability, and $sp = \mu \circ \rho$.*

1. *sp satisfies properties (1) and (2) of Theorem 11.4*
2. *If Ψ is a σ -semilattice and if for all ψ_1, ψ_2, \dots*

$$\rho(\bigvee_{i=1}^{\infty} \psi_i) = \bigwedge_{i=1}^{\infty} \rho(\psi_i),$$

then (3) of Theorem 11.4 holds.

3. *If Ψ is a complete lattice and if for any directed set $X \subseteq \Psi$*

$$\rho(\bigvee X) = \bigwedge_{\psi \in X} \rho(\psi),$$

then

$$sp(\bigvee X) = \inf_{\psi \in X} sp(\psi). \quad (11.5)$$

Proof. (1) and (2) are proved as in the proof of Theorem 11.4.

(3) The set $\{\rho(\psi) : \psi \in X\}$ is downwards directed if X is directed. Therefore, by Lemma 9.1

$$\mu(\rho(\bigvee X)) = \mu(\bigwedge_{\psi \in X} \rho(\psi)) = \inf_{\psi \in X} \mu(\rho(\psi)).$$

This proves (11.5). \square

Next, we consider *algebraic* information algebras $(\Psi, D; \leq, \perp, \cdot, \epsilon)$, with finite elements Ψ_f . We always suppose that $(\Psi_f, D; \leq, \perp, \cdot, \epsilon)$ is a subalgebra of $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ so that Ψ can be considered as the ideal completion I_{Ψ_f} of Ψ_f (see Theorem 7.5). In other words, the results to be derived below apply also to the ideal completion I_{Ψ} of any idempotent generalised information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. In this context we remind that a *generalised random variable* Γ is the supremum of the simple random variables it dominates, $\Gamma = \bigvee \{\Delta : \Delta \in \mathcal{R}_s, \Delta \leq \Gamma\}$. Simple random variables are here and in the sequel always assumed to take finite elements as values, that is $\Delta(\omega) \in \Psi_f$ for all ω . In such a case, the support function of a generalised random variable can be approximated by its values for finite elements.

Theorem 11.6 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be an algebraic information algebra, with Ψ_f as finite elements and Γ a generalised random variable with values in $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. Further let $sp_{\Gamma} = \mu \circ \rho_{\Gamma}$, where $\rho_{\Gamma} = \rho_0 \circ s_{\Gamma}$ (see (9.12)). Then for all $\psi \in \Psi$,*

$$sp_{\Gamma}(\psi) = \inf\{sp_{\Gamma}(\phi) : \phi \in \Psi_f, \phi \leq \psi\}. \quad (11.6)$$

Furthermore, if $X \subseteq \Psi$ is directed, then

$$sp_{\Gamma}(\bigvee X) = \inf_{\psi \in X} sp_{\Gamma}(\psi). \quad (11.7)$$

Proof. Note that (11.6) is a particular case of (11.7). By Theorem 10.10 we have $\rho_{\Gamma}(\bigvee X) = \bigwedge_{\psi \in X} \rho_{\Gamma}(\psi)$. Then (11.7) follows from Theorem 11.5 (3). \square

In the same framework, if $\Gamma = \bigvee_{i=1}^{\infty} \Delta_i$ is a *random variable* defined by a sequence family of simple random variables $\Delta_1, \Delta_2, \dots$, then the degree of support of any element in $\sigma(\Psi_f)$ may be obtained as a limit of the degrees of support of finite elements. In fact, if $\psi \in \sigma(\Psi_f)$, then $\psi = \bigvee_{i=1}^{\infty} \psi_i$, where $\psi_i \in \Psi_f$ (Theorem 9.3). We may always assume that the sequence ψ_i is monotone, $\psi_1 \leq \psi_2 \leq \dots$. Then this sequence is a directed set in Ψ and Theorem 11.6 applies. But due to the monotonicity of the sequence, we have $\inf\{sp_{\Gamma}(\psi_i) : i = 1, 2, \dots\} = \lim_{i \rightarrow \infty} sp_{\Gamma}(\psi_i)$. So, if $\psi = \bigvee_{i=1}^{\infty} \psi_i$ and $\psi_1 \leq \psi_2 \leq \dots \in \Psi_f$, then

$$sp_{\Gamma}(\psi) = \lim_{i \rightarrow \infty} sp_{\Gamma}(\psi_i). \quad (11.8)$$

The degree of support of a random variable can in some cases also be approximated by the degrees of support of the simple random variables which approximate the random variable.

Theorem 11.7 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be an idempotent generalised information algebra and $(\sigma(\Psi), D; \leq, \perp, \cdot, \epsilon)$ its σ -extension in I_{Ψ} . If $\Gamma = \bigvee_{i=1}^{\infty} \Delta_i$,*

where Δ_i are simple random variables with values in Ψ , is a random variable defined on the probability space (Ω, \mathcal{A}, P) with values in $\sigma(\Psi)$, then all elements $\psi \in \Psi$ are Γ -measurable, $\mathcal{E}_\Gamma = \Psi$. Furthermore, if the Δ_i form a monotone increasing sequence of simple random variables, then for all $\psi \in \Psi$,

$$sp_\Gamma(\psi) = \lim_{i \rightarrow \infty} sp_{\Delta_i}(\psi). \quad (11.9)$$

Proof. If Γ is a random variable defined by $\Gamma = \bigvee_{i=1}^{\infty} \Delta_i$, we may always assume that the Δ_i form a monotone sequence of simple random variables. Consider any $\psi \in \Psi$ and its support $s_\Gamma(\psi)$ relative to the random variable Γ . Then $\Delta_i \leq \Gamma$ implies $s_{\Delta_i}(\psi) \subseteq s_\Gamma(\psi)$, hence $\bigcup_{i=1}^{\infty} s_{\Delta_i}(\psi) \subseteq s_\Gamma(\psi)$. On the other hand we have

$$s_\Gamma(\psi) = \{\omega \in \Omega : \psi \leq \bigvee_{i=1}^{\infty} \Delta_i(\omega)\}.$$

Consider an $\omega \in s_\Gamma(\psi)$. As a monotone sequence, the $\Delta_i(\omega)$ form a directed set. Its supremum $\Gamma(\omega)$ belongs to the algebraic information algebra $(I_\Psi, D; \leq, \perp, \cdot, \epsilon)$, whose finite elements are given by Ψ . Therefore, by compactness, there must be an index i such that $\psi \leq \Delta_i(\omega)$, hence $\omega \in s_{\Delta_i}(\psi)$. But this shows that $s_\Gamma(\psi) \subseteq \bigcup_{i=1}^{\infty} s_{\Delta_i}(\psi)$, hence

$$s_\Gamma(\psi) = \bigcup_{i=1}^{\infty} s_{\Delta_i}(\psi). \quad (11.10)$$

Now, $s_{\Delta_i}(\psi)$ is measurable for all i , hence $s_\Gamma(\psi)$ is so too. This proves the first part of the theorem.

If the sequence of the Δ_i is monotone increasing, then so is $s_{\Delta_i}(\psi)$ for any $\psi \in \Psi$. Then (11.9) follows from (11.10) and the continuity of probability. \square

Another approximation of degrees of support by the degrees of support of simple random variables can be stated for generalised random variables.

Corollary 11.1 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be an idempotent generalised information algebra and Γ a generalised random variable in Ψ . Then, for all $\psi \in \Psi$,*

$$sp_\Gamma(\psi) = \sup\{sp_\Delta(\psi) : \Delta \in \mathcal{R}_s, \Delta \leq \Gamma\}. \quad (11.11)$$

Proof. We have by Theorem 10.8 that

$$\rho_\Gamma(\psi) = \bigvee\{\rho_\Delta(\psi) : \Delta \leq \Gamma\}.$$

Here, as in the sequel, Δ always denote simple random variables. Let (μ, \mathcal{B}) be the probability algebra associated with the probability space on which Γ is

defined. Then $sp_\Gamma = \mu \circ \rho_\Gamma$. The set $\{\rho_\Delta(\psi) : \Delta \leq \Gamma\}$ is downwards directed in \mathcal{B} . Therefore, by Lemma 9.1, we conclude that $sp_\Gamma(\psi) = \mu(\rho_\Gamma(\psi)) = \sup\{\mu(\rho_\Delta(\psi)) : \Delta \leq \Gamma\} = \sup\{sp_\Delta(\psi) : \Delta \leq \Gamma\}$. \square

We are in this chapter going to study functions monotone of order ∞ , satisfying properties (1) and (2) from Theorem 11.4 above. As we have seen, such functions do arise from random mappings in different ways and also from allocations of probability. Therefore, we define a corresponding class of functions.

Definition 11.2 *Let \mathcal{E} be a join-semilattice with a least element 1. Then a function $sp : \mathcal{E} \rightarrow [0,1]$ satisfying (1) and (2) below is called a support function on \mathcal{E} :*

1. $sp(1) = 1$.
2. If $\psi_1, \dots, \psi_m \geq \psi$, $\psi_1, \dots, \psi_m, \psi \in \mathcal{E}$,

$$sp(\psi) \geq \sum_{\emptyset \neq I \subseteq \{1, \dots, m\}} (-1)^{|I|+1} sp(\bigvee_{i \in I} \psi_i). \quad (11.12)$$

3. If in addition \mathcal{E} is closed under countable joins, and for any montone sequence $\psi_1 \leq \psi_2 \leq \dots$ the condition

$$sp\left(\bigvee_{i=1}^{\infty} \psi_i\right) = \lim_{i \rightarrow \infty} sp(\psi_i) \quad (11.13)$$

holds, then sp is called a continuous support function of \mathcal{E} .

4. If further \mathcal{E} is a complete semilattice and for any directed set $X \subseteq \mathcal{E}$,

$$sp\left(\bigvee X\right) = \inf_{\psi \in X} sp(\psi) \quad (11.14)$$

holds, then sp is called a condensable support function on \mathcal{E} .

So, for any random mapping Γ , the function sp_Γ is a *support function* on \mathcal{E}_Γ . Random variables Γ have *continuous* support functions sp_Γ and the support functions $sp_\Gamma = \mu \circ \rho_\Gamma$ of generalised random variables Γ are *condensable* on Ψ , if $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is an algebraic information algebra. We are going to study such support functions. The first question we are going to examine, is whether any support function can be obtained as the support function of a random mapping. This question will be addressed in the next section. Further, if a support function is defined on some sub-semilattice \mathcal{E} of an information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$, how can this function be extended to all of Ψ ? This question will be studied in Section 11.3.

11.2 Generating Support Functions

Any random mapping Γ from some probability space (Ω, \mathcal{A}, P) into an information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ generates a support function sp_Γ on the join-semilattice $\mathcal{E}_\Gamma \subseteq \Psi$ of its Γ -measurable elements. We remind that \mathcal{E}_Γ contains at least the element 1 of Ψ . Now, suppose that \mathcal{E} is a join-semilattice containing a least element 1 and that $sp : \mathcal{E} \rightarrow \mathbb{R}$ is a support function according to Definition 11.2 in the previous section. In fact, we shall always consider \mathcal{E} as a sub-semilattice of some idempotent generalised information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. Is there a random mapping Γ into Ψ such that its support function sp_Γ coincides with sp on \mathcal{E} ? We show in this section that the answer is affirmative, with the small amendment, that the mapping is into the ideal completion I_Ψ of Ψ rather than into Ψ itself.

This result is based on the *Theorem of Krein-Milman* which states that in a locally convex topological space which is Hausdorff, any compact convex set S is the closure of the convex hull of its extreme points (Phelps, 2001). The set S consists in our case of the support functions in the space of real-valued functions on \mathcal{E} . We shall use a result of Choquet on the extreme points of monotone functions of order ∞ (Choquet, 1953–1954). In fact, the theory presented here can be seen as part of Choquet's theory of capacities, and illustrates in particular the connection of capacities to probability.

Let \mathcal{E} be a join-semilattice, containing the least element 1. Consider the vector space V of functions $f : \mathcal{E} \rightarrow \mathbb{R}$ with pointwise addition and scalar multiplication. It becomes a topological space with pointwise convergence. Since \mathbb{R} is Hausdorff, so is V (Kelley, 1955). Define $p_\psi(f) = |f(\psi)|$ for $f \in V$ and $\psi \in \mathcal{E}$. Then p_ψ is a *semi-norm*, that is

1. it is *positive semidefinite*: $p_\psi(f) \geq 0$ for all $f \in V$,
2. it is *positive homogeneous*: $p_\psi(\lambda \cdot f) = \lambda \cdot p_\psi(f)$, for all $\lambda \geq 0$,
3. and it satisfies the *triangle inequality*: $p_\psi(f + g) \leq p_\psi(f) + p_\psi(g)$.

Therefore, V is a *locally convex* topological Hausdorff space.

Now, let S denote the set of all support functions on \mathcal{E} , which is a subset of V . The set S is obviously *convex* and *closed* in V . Furthermore, S is contained in the product space $\mathbb{R}^\mathcal{E} = \prod\{\mathbb{R} : \psi \in \mathcal{E}\}$. Define $S[\psi] = \{f(\psi) : f \in S\}$. These sets are *bounded* for all $\psi \in \mathcal{E}$ and their closures $\bar{S}[\psi]$ are therefore *compact*. By Tychonov's theorem (Kelley, 1955) the product $\prod\{\bar{S}[\psi] : \psi \in \mathcal{E}\}$ is compact and since $S \subseteq \prod\{\bar{S}[\psi] : \psi \in \mathcal{E}\}$, S is *compact* too.

Next we are going to apply the Krein-Milman theorem to the convex, compact set S . Here is the theorem:

Theorem 11.8 Theorem of Krein-Milman: *A convex, compact subset S of a locally convex Hausdorff space is the closed convex hull of its extreme points.*

Before we are going to apply this theorem to our problem of finding a random mapping inducing a given support function, we transform the theorem into an integral representation, following (Phelps, 2001). As a preparation we need a further notion. Let P be a probability measure on S , that is, a nonnegative regular measure on the σ -algebra of Borel sets in S , such that $P(S) = 1$. A point $f \in V$ is said to be represented by P , if for every linear function $h : V \rightarrow \mathbb{R}$,

$$h(f) = \int_S h(v) dP(v).$$

We cite the following lemma from (Phelps, 2001):

Lemma 11.1 *Let C be a compact subset of a locally convex topological space V . A point $f \in V$ belongs to the closed convex hull H of C , if and only if there is a probability measure P on C which represents f .*

Now, with the aid of this lemma, we reformulate the Krein-Milman Theorem 11.8.

Theorem 11.9 *Every point f of a convex, compact subset S of a locally convex Hausdorff space V is represented by a probability measure on S , which is supported by the closure of the extreme points $\text{ext}(S)$ of S , i.e. $P(\bar{\text{ext}}(S)) = 1$.*

Proof. By the Krein-Milman Theorem 11.8, $f \in S$ means, that f belongs to the closure of the convex hull of the extreme points $\text{ext}(S)$ of S . Clearly, the set of extreme point of S is bounded, its closure is therefore compact. Hence, by Lemma 11.1, f is represented by a probability on the closure of the extreme points of S . \square

What are the extreme points of the set S of support functions? This question is answered by Theorem 43.4 in (Choquet, 1953–1954). In this theorem Choquet considers functions *alternating of order ∞* . This means that in (11.12) of Definition 11.2 the inverse inequality holds. Now, if f is *monotone* of order ∞ , then $g(\psi) = f(1) - f(\psi)$ is *alternating* of order ∞ . So there is a close relation between the two notions. Choquet further considers alternating functions on an ordered commutative semigroup with a zero-element with all elements greater than zero. This applies to our join-semigroup \mathcal{E} , which, in addition, is an *idempotent* semigroup. If \mathcal{C} is a convex cone in V and \mathcal{H} is an affine subspace of V , not containing the zero function, and which meets every ray of \mathcal{C} , then $\mathcal{C} \cap \mathcal{H}$ is a convex set and $f \in \mathcal{C} \cap \mathcal{H}$ is an extreme point of this convex set, if and only if f is an extremal point of the convex cone \mathcal{C} . As a consequence of Theorem 43.4, Choquet states in Section 46 of (Choquet, 1953–1954) that the extremal points of the convex

cone \mathcal{M} of functions monotone to the order ∞ are the exponentials on \mathcal{E} , that is functions $e : \mathcal{E} \rightarrow \mathbb{R}$ such that $0 \leq e(\psi) \leq 1$, for all $\psi \in (E)$ and

$$e(\phi \cdot \psi) = e(\psi) \times e(\psi).$$

for all $\phi, \psi \in \mathcal{E}$.

Note now that item 1 of Definition 11.2 requires for a support function that $f(1) = 1$. This defines an affine hyperplane \mathcal{H} in V and $\mathcal{M} \cap \mathcal{H}$ is exactly the set of support functions on \mathcal{E} . So its extreme points are the exponentials e on \mathcal{E} with $e(1) = 1$. Since \mathcal{E} is idempotent, we have for any exponential $e(\psi) = e(\psi \cdot \psi) = e(\psi) \times e(\psi)$. Hence $e(\psi)$ takes only the values 0 or 1. Let e_i for $i = 1, 2, \dots$ be a convergent sequence of exponentials on \mathcal{E} , such that

$$e(\psi) = \lim_{i \rightarrow \infty} e_i(\psi).$$

Then e is a support function, since S is closed, and its is also an exponential on \mathcal{E} . So the set of exponentials is both bounded and closed, hence compact. Define for an exponential e

$$I_e = \{\psi \in \mathcal{E} : e(\psi) = 1\}.$$

This is obviously an *ideal* in \mathcal{E} and any ideal I in \mathcal{E} defines an exponential by $e(\psi) = 1$ if $\psi \in I$ and $e(\psi) = 0$ otherwise. So, there is a one-to-one relation between exponentials on \mathcal{E} and ideals of \mathcal{E} . We may identify the set of exponentials on \mathcal{E} by the set $I_{\mathcal{E}}$ of ideals in \mathcal{E} .

Fix a $\psi \in \mathcal{E}$. Define, for $f \in V$, $h_{\psi}(f) = f(\psi)$. This defines a continuous linear function $h_{\psi} : V \rightarrow \mathbb{R}$. Consider now any support function $sp \in S$. By the reformulated version of the Krein-Milman Theorem, 11.9, sp is represented by a probability measure on the closed set of its extreme points, that is, the set of exponentials on \mathcal{E} . Hence, we have

$$sp(\psi) = h_{\psi}(sp) = \int_{ext(S)} h_{\psi}(e) dP(e) = \int_{ext(S)} e(\psi) dP(e),$$

for some probability measure P supported by $ext(S)$ and all $\psi \in \mathcal{E}$. But, because e is a 0-1-function, this gives

$$sp(\psi) = P\{e : e(\psi) = 1\}.$$

Now, we are nearly done. We consider the probability space $(ext(S), \mathcal{B}, P)$, where \mathcal{B} denotes the Borel σ -algebra of subsets of $ext(S)$ and P the probability introduced above. We now construct a mapping from $ext(S)$ into $(I_{\Psi}, D; \leq, \perp, \cdot, \epsilon)$, the ideal extension of the information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. Since \mathcal{E} is supposed to be a sub-semilattice of Ψ , the ideal I_e associated with the exponential e can be extended to an ideal in Ψ , generally in many ways, for example by

$$J_e = \{\psi \in \Psi : \psi \leq \phi \text{ for some } \phi \in I_e\}.$$

Then we define the random mapping $\Gamma(e) = J_e$ from the probability space $(ext(S), \mathcal{B}, P)$ into the information algebra $(I_\Psi, D; \leq, \perp, \cdot, \epsilon)$. As usual, we consider Ψ as a subset of I_Ψ by the embedding $\psi \mapsto \downarrow \psi$. Let $\psi \in \mathcal{E}$. Then for the support of ψ by Γ we obtain

$$\begin{aligned} s_\Gamma(\psi) &= \{e \in ext(S) : \psi \in J_e\} = \{e \in ext(S) : \psi \in I_e\} \\ &= \{e \in ext(S) : e(\psi) = 1\}. \end{aligned}$$

As we have seen, the last set is measurable, that is belongs to \mathcal{B} . Hence we see that all elements of \mathcal{E} are Γ -measurable, $\mathcal{E} \subseteq \mathcal{E}_\Gamma$. Further,

$$sp_\Gamma(\psi) = P(s_\Gamma(\psi)) = P\{e \in ext(S) : e(\psi) = 1\} = sp(\psi).$$

So sp_Γ and sp coincide on \mathcal{E} . In this sense sp is induced by the random mapping Γ , hence Γ generates sp . We should stress that the Γ defined above is not the unique random mapping generating sp . This issue will be addressed in Section 11.3.

Next we turn to *continuous* support functions. This time let \mathcal{E} be a σ -join-semilattice, a semilattice closed under *countable* joins. Again, we assume \mathcal{E} to be a sub-semilattice of some σ -information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. Let S_c denote the set of continuous support functions on \mathcal{E} . As above, we argue that S_c is still a convex, compact subset of the function space V . Therefore, the revised Theorem of Krein-Milman 11.9 still applies. Because the elements of S_c are still monotone of order ∞ , Choquet's Theorem 43.4 (Choquet, 1953–1954) is also still applicable. The extreme elements of S_c are therefore again *exponentials* on \mathcal{E} . But since they belong to S_c , they must be *continuous* exponentials. That is, if $\psi_1 \leq \psi_2 \leq \dots$ is a monotone sequence in \mathcal{E} , then

$$e\left(\bigvee_{i=1}^{\infty} \psi_i\right) = \lim_{i \rightarrow \infty} e(\psi_i).$$

Since e is a monotone 0-1 function it follows that

$$e\left(\bigvee_{i=1}^{\infty} \psi_i\right) = \prod_{i=1}^{\infty} e(\psi_i).$$

The set of extreme points $ext(S_c)$ is again bounded and closed, hence compact. As above, define $I_e = \{\psi \in \mathcal{E} : e(\psi) = 1\}$. This time I_e becomes a σ -ideal in \mathcal{E} .

Consider a continuous support function $sp \in S_c$. Define, as above, $h_\psi(f) = f(\psi)$, a linear function from V into \mathbb{R} . By Theorem 11.9 there exists a probability measure P on $ext(S_c)$ such that

$$sp(\psi) = h_\psi(sp) = \int_{ext(S_c)} h_\psi(e) dP(e) = \int_{ext(S_c)} e(\psi) dP(e).$$

As above this gives

$$sp(\psi) = P\{e \in ext(S_c) : e(\psi) = 1\},$$

So, again as above, we may define a random mapping from the probability space $(ext(S_c), \mathcal{B}_c, P)$ into the ideal completion $(I_\Psi, D; \leq, \perp, \cdot, \epsilon)$ of the information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$, by $\Gamma(e) = J_e$. Here \mathcal{B}_c is the σ -field of Borel sets in $ext(S_c)$. Note that in this case J_e is a σ -ideal in Ψ . As above we verify that

$$sp_\Gamma(\psi) = P(s_\Gamma(\psi)) = P\{e \in ext(S_c) : e(\psi) = 1\} = sp(\psi)$$

for all $\psi \in \mathcal{E}$. So, Γ is a random mapping generating the continuous support function sp on \mathcal{E} .

To conclude this part, we formulate the main result of this section in the following theorem

Theorem 11.10 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be an idempotent generalised information algebra and $\mathcal{E} \subseteq \Psi$ a join-sub-semilattice of Ψ containing 1. If sp is a support function on \mathcal{E} , then there exists a probability space (Ω, \mathcal{A}, P) and a random mapping Γ from this space into the ideal completion of $(I_\Psi, D; \leq, \perp, \cdot, \epsilon)$ of $(\Psi, D; \leq, \perp, \cdot, \epsilon)$, such that $\mathcal{E} \subseteq \mathcal{E}_\Gamma$ and its support function coincides on \mathcal{E} , with sp , that is $sp_\Gamma(\psi) = sp(\psi)$ for all $\psi \in \mathcal{E}$.*

If $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is a σ -information algebra, $\mathcal{E} \subseteq \Psi$ a σ -semilattice and sp continuous, then there is a random mapping Γ generating sp , as in the first part of the theorem, which maps to σ -ideals of Ψ .

We remark that for continuous support functions there is an alternative approach to generate it from a random mapping, due to (Norberg, 1989).

11.3 Canonical Random Mappings

According to the previous Section 11.2 any support function can be generated by some random mapping. In this section we are going to examine the random mappings generating a given support function in more detail. In particular, we shall compare these random mappings and single out a particular one, which we shall call the *canonical* mapping.

Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be an information algebra and $\mathcal{E} \subseteq \Psi$ a join-sub-semilattice of Ψ , containing 1. Consider a support function sp on \mathcal{E} . According to the discussion in Section 11.2 there is a probability space $(ext(S), \mathcal{A}, P)$ on the set of exponentials $ext(S)$ on \mathcal{E} and a random mapping into the *ideal completion* of $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ defined by

$$\nu(e) = J_e = \{\psi \in \Psi : \psi \leq \phi \text{ for some } \phi \in I_e\}$$

where I_e is the ideal $\{\psi \in \mathcal{E} : e(\psi) = 1\}$ in \mathcal{E} associated with the exponential e . Then we obtain for $\psi \in \mathcal{E}$

$$sp(\psi) = P\{e \in ext(S) : e(\psi) = 1\},$$

which shows that the random mapping ν from $ext(S)$ into the ideal completion I_Ψ of Ψ indeed generates the support function on \mathcal{E} .

We noted in Section 11.2 that there is a one-to-one relation between exponentials $e \in ext(S)$ on \mathcal{E} and the ideals $I_\mathcal{E}$ in \mathcal{E} . To each exponential e corresponds the ideal I_e in \mathcal{E} and conversely, any ideal I of \mathcal{E} defines an exponential e_I by $e_I(\psi) = 1$, if $\psi \in I$ and $e_I(\psi) = 0$ otherwise. We may therefore replace the probability space $(ext(S), \mathcal{A}, P)$ on $ext(S)$ by an equivalent probability space $(I_\mathcal{E}, \mathcal{A}, P)$ on $I_\mathcal{E}$. By abuse of notation we denote here the σ fields and the probability measures in both spaces by the same symbol. The random mapping ν is then changed in the obvious way to

$$\nu(I) = \{\psi \in \Psi : \psi \leq \phi \text{ for some } \phi \in I\}.$$

We remarked in Section 11.2 that this random mapping ν is not the only one inducing the support function on \mathcal{E} . Let's examine this in more detail. The restriction of an ideal I of Ψ to \mathcal{E} is clearly an ideal of \mathcal{E} . We define the mapping $p : I_\Psi \rightarrow I_\mathcal{E}$ by $p(I) = I|_\mathcal{E} = I \cap \mathcal{E}$; to each ideal in Ψ , we associate its restriction to \mathcal{E} . Then the inverse mapping $p^{-1}(I) = \{J \in I_\Psi : p(J) = I\}$ induces a partition of I_Ψ . Consider any ideal $J \in p^{-1}(I)$. Obviously we have $\nu(I) \subseteq J$. Thus, $\nu(I)$ is the *least* ideal in $p^{-1}(I)$.

Consider any random mapping Γ from $I_\mathcal{E}$ into the ideal completion I_Ψ of Ψ , such that $\Gamma(I) \in p^{-1}(I)$. Its allocation of support is, for $\psi \in \mathcal{E}$,

$$s_\Gamma(\psi) = \{I \in I_\mathcal{E} : \psi \in \Gamma(I)\} = \{I \in I_\mathcal{E} : \psi \in I\}.$$

It follows that the random mapping Γ induces also the support function sp on \mathcal{E} ,

$$sp_\Gamma(\psi) = P(s_\Gamma(\psi)) = P\{I \in I_\mathcal{E} : \psi \in I\} = sp(\psi).$$

Hence, ν is the *minimal* random mapping on $I_\mathcal{E}$ generating sp .

Let's pursue this observation. Consider the probability algebra (\mathcal{B}, μ) associated with the probability space $(I_\mathcal{E}, \mathcal{A}, P)$ (see Section 9.2). We remind that the mapping $\rho_\nu = \rho_0 \circ s_\nu$ from Ψ into \mathcal{B} is an allocation of probability (a.o.p) (see Section 9.2). This a.o.p, as every a.o.p on Ψ , induces a support function $sp_\nu = \mu \circ \rho_0 \circ s_\nu$ on Ψ (see Theorem 11.5), and its restriction to \mathcal{E} equals sp . So, sp_ν is an extension of sp to Ψ . Now, for any random mapping Γ from $I_\mathcal{E}$ into the ideal completion I_Ψ of Ψ , such that $\Gamma(I) \in p^{-1}(I)$, we have $\nu(I) \subseteq \Gamma(I)$. This implies for the allocations of support that $s_\nu(\psi) \subseteq s_\Gamma(\psi)$, hence $\rho_\nu(\psi) = \rho_0(s_\nu(\psi)) \leq \rho_0(s_\Gamma(\psi)) = \rho_\Gamma(\psi)$ and for $\psi \in \mathcal{E}$, we have $\rho_\nu(\psi) = \rho_\Gamma(\psi)$. It follows that

$$sp_\nu(\psi) = \mu(\rho_0(s_\nu(\psi))) \leq \mu(\rho_0(s_\Gamma(\psi))) = sp_\Gamma(\psi)$$

We shall see later (Section 11.4) that the random mapping ν generates indeed the *least* extension of the support function sp on \mathcal{E} to Ψ among all extensions. But before we turn to this question, we return to the random mappings generating sp on \mathcal{E} .

Consider the family of sets $\{I \in I_{\mathcal{E}} : \psi \in I\}$ for $\psi \in \mathcal{E}$. All these sets belong to the σ -field \mathcal{A} in the probability space $(I_{\mathcal{E}}, \mathcal{A}, P)$ used to define the random mapping ν to generate the support function sp on \mathcal{E} and $sp(\psi) = P(I \in I_{\mathcal{E}} : \psi \in I)$. Let $\mathcal{A}_{\mathcal{E}} \subseteq \mathcal{A}$ be the σ -field of subsets generated by the family of these subsets. Note that this set depends *only* on the semi lattice \mathcal{E} , but not on sp itself. Denote the restriction of the probability measure P to $\mathcal{A}_{\mathcal{E}}$ by P_{sp} . This probability depends on the support function sp , and thereby indirectly of course also on \mathcal{E} . Consider the probability space $(I_{\mathcal{E}}, \mathcal{A}_{\mathcal{E}}, P_{sp})$. We remark that the random mapping ν , as well as the related mappings Γ considered above, still generate sp on \mathcal{E} .

In order to facilitate comparisons between random mappings generating the support function sp on \mathcal{E} , we transport probability from the set of ideal $I_{\mathcal{E}}$ in \mathcal{E} to the set I_{Ψ} of ideals in Ψ . The family of sets $p^{-1}(A)$ for $A \in \mathcal{A}_{\mathcal{E}}$ forms a σ -field of subsets of I_{Ψ} and by $P(p^{-1}(A)) = P_{sp}(A)$ a probability measure is defined on this σ -field. By abuse of notation, we denote the new probability space by $(I_{\Psi}, \mathcal{A}_{\mathcal{E}}, P_{sp})$. The random mapping ν from $I_{\mathcal{E}}$ into the ideal completion of Ψ is redefined as $\nu(p(I))$ for $I \in I_{\Psi}$. Again, we call this new mapping ν , that is,

$$\nu(I) = \{\psi \in \Psi : \phi \leq \psi \text{ for some } \phi \in p(I)\}. \quad (11.15)$$

We call this random mapping ν , together with the associated probability space $(I_{\Psi}, \mathcal{A}_{\mathcal{E}}, P_{sp})$, the *canonical random mapping* generating the support function sp on the semilattice \mathcal{E} . Any other random mapping Γ defined above on $I_{\mathcal{E}}$ may similarly be redefined as $\Gamma(p(I))$.

We can now compare different extensions of support functions from \mathcal{E} . Consider semilattices \mathcal{E}_1 and \mathcal{E}_2 such that $\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \Psi$ and support functions sp_1 and sp_2 on \mathcal{E}_1 and \mathcal{E}_2 respectively, such that sp_2 is an extension of sp_1 . Then, these support functions have their canonical random mappings ν_1 and ν_2 defined on the probability spaces $(I_{\Psi}, \mathcal{A}_{\mathcal{E}_1}, P_{sp_1})$ and $(I_{\Psi}, \mathcal{A}_{\mathcal{E}_2}, P_{sp_2})$ respectively. The next theorem shows how these canonical random mappings are related.

Theorem 11.11 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be an idempotent generalised information algebra and let ν_1 and ν_2 , defined on the probability spaces $(I_{\Psi}, \mathcal{A}_{\mathcal{E}_1}, P_{sp_1})$ and $(I_{\Psi}, \mathcal{A}_{\mathcal{E}_2}, P_{sp_2})$, be the canonical random mappings associated with the support functions sp_1 and sp_2 on the semilattices $\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \Psi$. If sp_2 is an extension of sp_1 , that is $sp_1 = sp_2|_{\mathcal{E}_1}$, then*

1. $\nu_1 \leq \nu_2$, in the order of the information algebra of random mappings into $(I_{\Psi}, D; \leq, \perp, \cdot, \epsilon)$,

2. $\mathcal{A}_{\mathcal{E}_1} \subseteq \mathcal{A}_{\mathcal{E}_2}$,
3. $P_{sp_1} = P_{sp_2}|_{\mathcal{A}_{\mathcal{E}_1}}$, on $\mathcal{A}_{\mathcal{E}_1}$ the two probability measures are equal.
4. $sp_{\nu_1}(\psi) \leq sp_{\nu_2}(\psi)$ for all $\psi \in \Psi$.

Proof. (1) By definition we have $p_1(I) = I|\mathcal{E}_1$ and $p_2(I) = I|\mathcal{E}_2$, hence $p_1(I) \subseteq p_2(I)$. Therefore, from (11.15), we conclude that $\nu_1(I) \subseteq \nu_2(I)$ for all $I \in I_\Psi$, hence $\nu_1 \leq \nu_2$.

(2) Consider an element $\psi \in \mathcal{E}_1 \subseteq \mathcal{E}_2$. Then, the allocations of support relative to ν_1 and ν_2 , respectively, are

$$\begin{aligned} s_{\nu_1}(\psi) &= \{I \in I_\Psi : \psi \in \nu_1(I)\} = \{I \in I_\Psi : \psi \in I|\mathcal{E}_1\}, \\ s_{\nu_2}(\psi) &= \{I \in I_\Psi : \psi \in \nu_2(I)\} = \{I \in I_\Psi : \psi \in I|\mathcal{E}_2\}. \end{aligned}$$

But $\psi \in I|\mathcal{E}_1$ implies $\psi \in I|\mathcal{E}_2$. On the other hand, $\psi \in \mathcal{E}_1$ and $\psi \in I|\mathcal{E}_2$ implies $\psi \in I|\mathcal{E}_2 \cap \mathcal{E}_1 = I|\mathcal{E}_1$. So, we conclude that $s_{\nu_1}(\psi) = s_{\nu_2}(\psi)$ for every $\psi \in \mathcal{E}_1$. Since $\mathcal{A}_{\mathcal{E}_1}$ is the σ -field generated by the allocations $s_{\nu_1}(\psi)$ for $\psi \in \mathcal{E}_1$, and $\mathcal{A}_{\mathcal{E}_2}$ the one generated by $s_{\nu_2}(\psi)$ for $\psi \in \mathcal{E}_2 \supseteq \mathcal{E}_1$, this shows that $\mathcal{A}_{\mathcal{E}_1} \subseteq \mathcal{A}_{\mathcal{E}_2}$.

(3) To prove this claim, we use Dynkin's Theorem (Billingsley, 1995). Dynkin calls a family of sets, closed under finite intersections, a π -system. The family P of sets $s_{\nu_1}(\psi)$ for $\psi \in \mathcal{E}_1$ is a π -system (see Theorem 11.1). The family L of sets $A \in \mathcal{A}_{\mathcal{E}_1}$ for which

$$P_{sp_1}(A) = P_{sp_2}(A)$$

is closed under complementation, and contains $\cup_i A_i$, if A_i is a countable family of disjoint sets in L . This is called a λ -system by Dynkin. From the considerations above, we conclude that $P \subseteq L$. The theorem of Dynkin states that if P is a π -system and L a λ -system, then $P \subseteq L$ implies that the σ -closure of P is contained in L , that is $\sigma(P) \subseteq L$. In our case the σ -closure of P is $\mathcal{A}_{\mathcal{E}_1}$, hence we have $\mathcal{A}_{\mathcal{E}_1} \subseteq L$, where L contains all sets of $\mathcal{A}_{\mathcal{E}_1}$ on which the two probabilities coincide. So, indeed for all $A \in \mathcal{A}_{\mathcal{E}_1}$ we have $P_{sp_1}(A) = P_{sp_2}(A)$.

(4) We have for any $\psi \in \Psi$ (see (9.15)) $sp_{\nu_1}(\psi) = P_{sp_1*}(s_{\nu_1}(\psi)) \leq P_{sp_2*}(s_{\nu_2}(\psi)) = sp_{\nu_2}(\psi)$, because $s_{\nu_1}(\psi) \subseteq s_{\nu_2}(\psi)$ and $P_{sp_1*}(A) \leq P_{sp_2*}(A)$ for any set $A \subseteq I_\Psi$. Therefore, $sp_{\nu_1}(\psi) \leq sp_{\nu_2}(\psi)$. \square

This theorem shows in particular, that the canonical random mapping associated with a support function sp on a semilattice $\mathcal{E} \subseteq \Psi$ is *unique*. It permits also to conclude that sp_ν is indeed the *least* extension of the support function sp from \mathcal{E} to Ψ . Indeed, suppose that sp' is any extension of sp to Ψ . Then, sp' is generated by a canonical random mapping ν' . According to Theorem 11.11 (4) we have then

$$sp_\nu(\psi) \leq sp_{\nu'}(\psi) = sp'(\psi).$$

The last equity holds because sp' is defined on Ψ . So, we have

Corollary 11.2 *If sp is a support function defined on a semilattice $\mathcal{E} \subseteq \Psi$, then sp_ν is the least extension of sp to Ψ , that is, $sp_\nu \leq sp'$ for any support function sp' on Ψ such that $sp = sp'|_{\mathcal{E}}$.*

We remark, that a similar analysis can be done for σ -semilattices or complete lattices \mathcal{E} and continuous or condensable support functions sp . However, more interesting is the case of *algebraic* information algebras $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. As usual, we assume that the finite elements $(\Psi_f, D; \leq, \perp, \cdot, \epsilon)$ form a subalgebra of $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. We consider a support function sp defined on Ψ_f , hence $\mathcal{E} = \Psi_f$. Since in this case its ideal completion $(I_{\Psi_f}, D; \leq, \perp, \cdot, \epsilon)$ is isomorphic to $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ we identify ideals I of Ψ_f with their suprema $\bigvee I \in \Psi$. For the support function sp , we consider its canonical probability space $(I_{\Psi_f}, \mathcal{A}_{\Psi_f}, P_{sp})$.

Instead of the canonical random mapping,

$$\nu(I) = \{\psi \in \Psi : \psi \leq \phi \text{ for some } \phi \in I\}$$

we consider also the random mappings

$$\sigma(I) = \{\psi \in \Psi : \psi \leq \bigvee_{i=1}^{\infty} \psi_i \text{ for some } \psi_i \in I\}, \quad (11.16)$$

$$\gamma(I) = \downarrow \bigvee I. \quad (11.17)$$

Both map I_{Ψ_f} into I_{Ψ} . We are going to examine the support functions on Ψ induced by these random mappings.

We start with the random mapping σ . Here are its basic properties:

Lemma 11.2 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be an algebraic information algebra, its finite elements $(\Psi_f, D; \leq, \perp, \cdot, \epsilon)$ a subalgebra and σ the random map defined by (11.16). Then for an ideal $I \in I_{\Psi_f}$,*

1. *the ideal $\sigma(I)$ is a σ -ideal in Ψ ,*
2. *its restriction to Ψ_f equals I , $\sigma(I)|_{\Psi_f} = I$,*
3. *the σ -ideal $\sigma(I)$ is minimal among all σ -ideals in Ψ extending I .*

Proof. (1) Assume $\psi_1, \psi_2, \dots \in \sigma(I)$. Then we have $\psi_i \leq \bigvee_{j=1}^{\infty} \psi_{i,j}$ with $\psi_{i,j} \in I$ for all $i = 1, 2, \dots$ and $j = 1, 2, \dots$. But then we obtain

$$\bigvee_{i=1}^{\infty} \psi_i \leq \bigvee_{i=1}^{\infty} \bigvee_{j=1}^{\infty} \psi_{i,j} = \bigvee_{h=1}^{\infty} \psi'_h,$$

where $\psi'_h = \bigvee_{i=1}^h \bigvee_{j=1}^i \psi_{i,j} \in I$. This shows that $\bigvee_{i=1}^{\infty} \psi_i \in \sigma(I)$, hence $\sigma(I)$ is indeed a σ -ideal in Ψ .

(2) Assume that $\psi \in \sigma(I)$ and $\psi \in \Psi_f$. Then $\psi \leq \bigvee_{i=1}^{\infty} \psi_i$, with $\psi_i \in I$ for $i = 1, 2, \dots$. By the usual transformation, we may always assume that $\psi_1 \leq \psi_2 \leq \dots$. This monotone sequence is a directed set in Ψ . By compactness there exists a ψ_i such that $\psi \leq \psi_i$. This shows that $\psi \in I$. But clearly, $I \subseteq \sigma(I)$, therefore we see that indeed the restriction of $\sigma(I)$ to Ψ_f equals I .

(3) Consider a σ -ideal J whose restriction to Ψ_f equals I . Assume $\psi \in \sigma(I)$. Then $\psi \leq \bigvee_{i=1}^{\infty} \psi_i$, with ψ_i in I , hence in J . But then $\bigvee_{i=1}^{\infty} \psi_i \in J$ since J is a σ -ideal, therefore $\psi \in J$. This shows that $\sigma(I) \subseteq J$. Hence $\sigma(I)$ is indeed minimal among the σ -ideals extending I . \square

The random map σ generates a support function $sp_{\sigma} = \mu \circ \rho_{\sigma}$ on Ψ , where as usual (μ, \mathcal{B}) is the probability algebra associated with the probability space $(I_{\Psi_f}, \mathcal{A}_{\Psi_f}, P_{sp})$, and $\rho_{\sigma} = \rho_0 \circ s_{\sigma}$. We are going to show that sp_{σ} is a *continuous* extension of sp . The key is the following lemma:

Lemma 11.3 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be an algebraic information algebra, its finite elements $(\Psi_f, D; \leq, \perp, \cdot, \epsilon)$ a subalgebra, σ the random map defined by (11.16), and s_{σ} the allocation of support for the random map σ . Then, if $\psi_i \in \Psi$ for $i = 1, 2, \dots$,*

$$s_{\sigma}\left(\bigvee_{i=1}^{\infty} \psi_i\right) = \bigcap_{i=1}^{\infty} s_{\sigma}(\psi_i).$$

Proof. Since Ψ is a complete lattice, $\bigvee_{i=1}^{\infty} \psi_i \in \Psi$, and

$$s_{\sigma}\left(\bigvee_{i=1}^{\infty} \psi_i\right) = \{I \in I_{\Psi_f} : \bigvee_{i=1}^{\infty} \psi_i \leq \bigvee_{i=1}^{\infty} \phi_i, \phi_i \in I\}.$$

If $I \in s_{\sigma}(\bigvee_{i=1}^{\infty} \psi_i)$, then clearly $I \in s_{\sigma}(\psi_i)$ for all $i = 1, 2, \dots$. Conversely, assume $I \in s_{\sigma}(\psi_i)$ for all $i = 1, 2, \dots$. Then we have $\psi_i \leq \bigvee_{j=1}^{\infty} \psi_{i,j}$ with $\psi_{i,j} \in I$. This implies in the same way as in the proof of Lemma 11.2 that $\bigvee_{i=1}^{\infty} \psi_i \in \sigma(I)$, hence $I \in s_{\sigma}(\bigvee_{i=1}^{\infty} \psi_i)$ and this proves the lemma. \square

As a consequence of this lemma, we find that

$$\begin{aligned} \rho_{\sigma}\left(\bigvee_{i=1}^{\infty} \psi_i\right) &= \rho_0\left(s_{\sigma}\left(\bigvee_{i=1}^{\infty} \psi_i\right)\right) = \rho_0\left(\bigcap_{i=1}^{\infty} s_{\sigma}(\psi_i)\right) \\ &= \bigwedge_{i=1}^{\infty} \rho_0(s_{\sigma}(\psi_i)) = \bigwedge_{i=1}^{\infty} \rho_{\sigma}(\psi_i). \end{aligned} \quad (11.18)$$

The allocation of probability ρ_{σ} is a σ -a.o.p. By Theorem 11.5 sp_{σ} is a *continuous* support function extending sp on Ψ_f to Ψ . Since $\sigma(I)$ is the least σ -ideal among all σ -ideals extending the ideal I of Ψ_f to Ψ , we conclude that sp_{σ} is also the minimal continuous support function among all continuous support functions sp extending sp from Ψ_f to Ψ ,

$$sp_{\sigma} \leq \tilde{sp}(\psi), \text{ if } \tilde{sp} \text{ continuous, } \tilde{sp}|_{\Psi_f} = sp$$

for all $\psi \in \Psi$.

Let's fix this result in the following theorem:

Theorem 11.12 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be an algebraic information algebra, whose finite elements $(\Psi_f, D; \leq, \perp, \cdot, \epsilon)$ form a subalgebra, sp a support function defined on Ψ_f and σ the random map defined by (11.16). Then, if (μ, \mathcal{B}) is the probability algebra associated with the canonical probability space $(I_{\Psi_f}, \mathcal{A}_{\Psi_f}, P_{sp})$ and $\rho_\sigma = \rho_0 \circ s_\sigma$, then $sp_\sigma = \mu \circ \rho_\sigma$ is the minimal continuous extension of sp to Ψ among all continuous extensions.*

We turn to the random mapping γ , defined in (11.17). This mapping is characterised as follows:

Lemma 11.4 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be an algebraic information algebra, its finite elements $(\Psi_f, D; \leq, \perp, \cdot, \epsilon)$ a subalgebra and γ the random mapping defined by (11.17). Then the ideal $\gamma(I)$ is the minimal complete ideal in Ψ whose restriction to Ψ_f equals I , $\gamma(I)|\Psi_f = I$.*

Proof. We have $I \subseteq \downarrow \bigvee I \cap \Psi_f$. Consider then an element $\psi \in \downarrow \bigvee I \cap \Psi_f$. From $\psi \leq \bigvee I$ it follows, since I is a directed set, by compactness that there is a $\chi \in I$ such that $\psi \leq \chi$. But then $\psi \in \Psi_f$ implies $\psi \in I$. This proves that $\gamma(I)|\Psi_f = I$.

As a principal ideal in a complete lattice, $\gamma(I)$ is a complete ideal. Consider any other complete ideal J , whose restriction to Ψ_f equals I . But then $\bigvee I \leq \bigvee J$ and $J = \downarrow \bigvee J$, hence $\gamma(I) \subseteq J$. This proves the minimality of $\gamma(I)$. \square

Consider now *simple* random variables Δ on the canonical probability space $(I_{\Psi_f}, \mathcal{A}_{\Psi_f}, P_{sp})$. Any such random variable is defined by a measurable partition $B_i \in \mathcal{A}_{\Psi_f}$, $i = 1, \dots, m$ of I_{Ψ_f} and $\Delta(I) = \psi_i \in \Psi_f$ if $I \in B_i$. Note that $\Delta \leq \gamma$ if and only if $\psi_i \leq \bigvee I$ for $I \in B_i$ and $i = 1, \dots, m$. This leads to the following result:

Lemma 11.5 *The random mapping γ defined by (11.17) is a generalised random variable,*

$$\gamma = \bigvee \{ \Delta : \Delta \text{ simple random variable}, \Delta \leq \gamma \}.$$

Proof. We claim that for all $I \in I_{\Psi_f}$ we have $\gamma(I) = \bigvee \{ \Delta(I) : \Delta \leq \gamma \}$ where it is understood that Δ denotes a simple random variable. Clearly $\gamma(I) \geq \bigvee \{ \Delta(I) : \Delta \leq \gamma \}$. To prove the converse inequality, consider $I \in I_{\Psi_f}$. Then we have by density $\gamma(I) = \downarrow \bigvee \{ \psi \in \Psi_f : \psi \leq \bigvee I \}$. But $\psi \leq \bigvee I$ implies that there is a $\phi \in I$ such that $\psi \leq \phi$. But then $\phi \in \Psi_f$ implies that $\psi \in I$. So, $\psi \in I$ if and only if $\psi \leq \bigvee I$ and $\psi \in \Psi_f$. Define, for a $\psi \in I$,

$$\Delta_\psi(I) = \begin{cases} \psi & \text{if } \psi \in I, \\ 1 & \text{otherwise.} \end{cases}$$

The set $\{I : \psi \in I\}$ is measurable (belongs to \mathcal{A}_{Ψ_f}), hence Δ_ψ is a simple random variable and $\Delta_\psi(I) \leq \bigvee I$, hence $\Delta_\psi \leq \gamma$. Thus, we conclude

$$\gamma(I) = \downarrow \bigvee \{\Delta_\psi(I) : \psi \in I\} \leq \downarrow \bigvee \{\Delta(I) : \Delta \leq \gamma\}.$$

This proves the identity $\gamma(I) = \downarrow \bigvee \{\Delta(I) : \Delta \leq \gamma\}$, hence the lemma. \square

From this lemma it follows according to Theorem 10.10 that for a directed subset X of Ψ

$$\rho_\gamma(\bigvee X) = \bigwedge_{\psi \in X} \rho_\gamma(\psi).$$

Further, from Theorem 11.5 it follows that

$$sp_\gamma(\bigvee X) = \inf_{\psi \in X} sp(\psi).$$

This implies also that for any $\psi \in \Psi$,

$$sp_\gamma(\psi) = \inf\{sp(\phi) : \phi \in \Psi_f, \phi \leq \psi\}. \quad (11.19)$$

This means that sp_γ is the unique *condensable* extension of sp from Ψ_f to Ψ . We note also that according to Theorem 11.11, since $\nu \leq \sigma \leq \gamma$, we have $sp_\nu(\psi) \leq sp_\sigma(\psi) \leq sp_\gamma(\psi)$. These results (Theorem 11.12 and (11.19)) partly answer an open question posed in (Shafer, 1979). In this work it was shown that continuous and condensable extensions always exist if \mathcal{E} is a subset lattice. Here it is shown that they always exist if \mathcal{E} corresponds to the finite elements of a compact information algebra, independently whether Ψ_f is a lattice or not.

We summarise these results in the following theorem.

Theorem 11.13 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be an algebraic information algebra, whose finite elements Ψ_f form a subalgebra, sp a support function defined on Ψ_f and γ the random map defined by (11.17). If (μ, \mathcal{B}) is the probability algebra associated with the canonical probability space $(I_{\Psi_f}, \mathcal{A}_{\Psi_f}, P_{sp})$ and if $\rho_\gamma = \rho_0 \circ s_\gamma$, then $sp_\gamma = \mu \circ \rho_\gamma$ is the unique condensable extension of sp to Ψ .*

We conclude by proving the converse of Theorem 11.5 and thus characterising continuous and condensable support functions by their associated allocations of support.

Theorem 11.14 *1. If Ψ is a σ -semilattice, then $sp = \mu \circ \rho$ is continuous on Ψ if and only if ρ is a σ -allocation of probability, that is for $\psi_i \in \Psi$, $i = 1, 2, \dots$*

$$\rho(\bigvee_{i=1}^{\infty} \psi_i) = \bigwedge_{i=1}^{\infty} \rho(\psi_i). \quad (11.20)$$

2. If Ψ is a complete lattice, then $sp = \mu \circ \rho$ is condensable on Ψ if and only if for any directed set $X \subseteq \Psi$,

$$\rho(\bigvee X) = \bigwedge_{\psi \in X} \rho(\psi). \quad (11.21)$$

Proof. The if-part of both parts is already proved in Theorem 11.5, it remains thus only to prove the only-if-part

(1) Consider a countable set of elements $\psi_1, \psi_2, \dots \in \Psi$. We may always replace this sequence by a monotone sequence $\psi'_1 \leq \psi'_2 \leq \dots$ having the same supremum, $\bigvee_{i=1}^{\infty} \psi_i = \bigvee_{i=1}^{\infty} \psi'_i$, by defining $\psi'_i = \bigvee_{j=1}^i \psi_j$. Then $\rho(\psi'_1) \geq \rho(\psi'_2) \geq \dots$ is downwards directed. Therefore, by the continuity of sp and Lemma 9.1,

$$\begin{aligned} sp(\bigvee_{i=1}^{\infty} \psi_i) &= sp(\bigvee_{i=1}^{\infty} \psi'_i) = \lim_{i \rightarrow \infty} sp(\psi'_i) \\ &= \lim_{i \rightarrow \infty} \mu(\rho(\psi'_i)) = \mu(\bigwedge_{i=1}^{\infty} \rho(\psi'_i)) = \mu(\bigwedge_{i=1}^{\infty} \rho(\psi_i)). \end{aligned}$$

From $sp(\bigvee_{i=1}^{\infty} \psi_i) = \mu(\rho(\bigvee_{i=1}^{\infty} \psi_i))$ it follows that $\mu(\rho(\bigvee_{i=1}^{\infty} \psi_i)) = \mu(\bigwedge_{i=1}^{\infty} \rho(\psi_i))$. Since $\bigwedge_{i=1}^{\infty} \rho(\psi_i) \geq \rho(\bigvee_{i=1}^{\infty} \psi_i)$ and μ is a positive measure, it follows indeed that $\bigwedge_{i=1}^{\infty} \rho(\psi_i) = \rho(\bigvee_{i=1}^{\infty} \psi_i)$.

(2) Let $X \subseteq \Psi$ be directed. By the condensability of sp we obtain

$$\mu(\rho(\bigvee X)) = sp(\bigvee X) = \inf_{\psi \in X} sp(\psi) = \inf_{\psi \in X} \mu(\rho(\psi)).$$

Since the set $\{\rho(\psi) : \psi \in X\}$ is downwards directed, we get by Lemma 9.1 $\inf_{\psi \in X} \mu(\rho(\psi)) = \mu(\bigwedge_{\psi \in X} \rho(\psi))$, hence $\mu(\rho(\bigvee X)) = \mu(\bigwedge_{\psi \in X} \rho(\psi))$. Since $\bigwedge_{\psi \in X} \rho(\psi) \geq \rho(\bigvee X)$, we conclude that $\bigwedge_{\psi \in X} \rho(\psi) = \rho(\bigvee X)$ \square

If $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is an algebraic information algebra and $sp = \mu \circ \rho$ condensable on Ψ , then (11.21) implies also that for all $\psi \in \Psi$

$$\rho(\psi) = \bigwedge \{\rho(\psi) : \psi \in \Psi_f, \phi \leq \psi\}.$$

We are going to study these different extensions of a support functions from a part of Ψ to the whole of Ψ in the next section from a different angle.

To conclude this section, consider an a.o.p ρ defined on Ψ . It is generated by some random mapping Γ into the ideal completion I_Ψ of Ψ . However, this map is not unique as we have seen. This confirms a former remark, that a random map Γ contains more information than its associated a.o.p ρ_Γ . This explains why the map $\Gamma \mapsto \rho_\Gamma$ is, in general, not a homomorphism (see the end of Section 10.2).

11.4 Minimal Extensions

In the previous section, we have found an extension sp_ν for any support function sp on some join-sub-semilattice \mathcal{E} of an information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ to the whole of the algebra. This extension is defined in terms of the canonical random mapping associated with sp . In this section, we shall show how the extension sp_ν and other extensions can be defined explicitly in terms of the support function sp on \mathcal{E} . The following theorem is an extension to information algebras of a result due to (Shafer, 1973) for set algebras.

Theorem 11.15 *If sp is a support function defined on a join-semilattice $\mathcal{E} \subseteq \Psi$, where $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is an idempotent, generalised information algebra, then*

$$sp_\nu(\phi) = \sup \left\{ \sum_{\emptyset \neq I \subseteq \{1, \dots, n\}} (-1)^{|I|+1} sp(\bigvee_{i \in I} \psi_i) \right\} \quad (11.22)$$

where the supremum is to be taken over all finite sets of elements $\psi_1, \dots, \psi_n \geq \phi$, $n = 1, 2, \dots$ with $\psi_1, \dots, \psi_n \in \mathcal{E}$.

Proof. Let f denote the function on the right hand side of (11.22). We remark that f is equal to sp on \mathcal{E} . Note also that f is less or at most equal to sp_ν , since the latter, as a support function on Ψ , is monotone of order ∞ . Therefore, it is sufficient to show that f is a support function on Ψ , because then, according to Corollary 11.2 it must be greater or equal to sp_ν , so that $sp_\nu = f$ as claimed.

In order to prove f to be a support function, we use, following (Shafer, 1973) allocations of probability. Let ρ be the allocation of support associated with the canonical random mapping generating sp , such that for $\psi \in \mathcal{E}$,

$$sp(\psi) = \mu(\rho(\psi)),$$

where μ is the probability of the probability algebra (\mathcal{B}, μ) associated with the probability space $(I_\Psi, \mathcal{A}_\mathcal{E}, P_{sp})$ of the canonical probability space of the support function sp on \mathcal{E} . Further, $\rho = \rho_0 \circ s_\nu$ (see Section 9.2). Define for $\phi \in \Psi$,

$$\bar{\rho}(\phi) = \bigvee \{\rho(\psi) : \psi \in \mathcal{E}, \phi \leq \psi\}. \quad (11.23)$$

We are going to show that $\bar{\rho}$ is an a.o.p on Ψ . Obviously, for $\psi \in \mathcal{E}$, we have $\bar{\rho}(\psi) = \rho(\psi)$, hence in particular $\bar{\rho}(1) = \top$. Consider $\phi_1, \phi_2 \in \Psi$. Then $\phi_1, \phi_2 \leq \phi_1 \cdot \phi_2$, hence $\bar{\rho}(\phi_1), \bar{\rho}(\phi_2) \geq \bar{\rho}(\phi_1 \cdot \phi_2)$ or $\bar{\rho}(\phi_1) \wedge \bar{\rho}(\phi_2) \geq \bar{\rho}(\phi_1 \cdot \phi_2)$. On the other hand, let $\phi_1 \leq \psi_1 \in \mathcal{E}$ and $\phi_2 \leq \psi_2 \in \mathcal{E}$. Then, $\psi_1 \cdot \psi_2 \in \mathcal{E}$

and $\phi_1 \cdot \phi_2 \leq \psi_1 \cdot \psi_2$ such that $\rho(\psi_1) \wedge \rho(\psi_2) = \rho(\psi_1 \cdot \psi_2) \leq \bar{\rho}(\psi_1 \cdot \psi_2)$. It follows that

$$\begin{aligned} \bar{\rho}(\phi_1 \cdot \phi_2) &\geq \bigvee \{ \rho(\psi_1) \wedge \rho(\psi_2) : \phi_1 \leq \psi_1, \phi_2 \leq \psi_2, \psi_1, \psi_2 \in \mathcal{E} \} \\ &= \left(\bigvee \{ \rho(\psi_1) : \phi_1 \leq \psi_1 \in \mathcal{E} \} \right) \wedge \left(\bigvee \{ \rho(\psi_2) : \phi_2 \leq \psi_2 \in \mathcal{E} \} \right) \\ &= \bar{\rho}(\phi_1) \wedge \bar{\rho}(\phi_2). \end{aligned}$$

So, we conclude that $\bar{\rho}(\phi_1 \cdot \phi_2) = \bar{\rho}(\phi_1) \wedge \bar{\rho}(\phi_2)$ and that, therefore, $\bar{\rho}$ is an a.o.p.

In the formula (11.22) for f , we may replace sp by $\mu \circ \rho$,

$$\begin{aligned} f(\phi) &= \sup \left\{ \sum_{\emptyset \neq I \subseteq \{1, \dots, n\}} (-1)^{|I|+1} \mu(\rho(\bigvee_{i \in I} \psi_i)) \right\} \\ &= \sup \left\{ \sum_{\emptyset \neq I \subseteq \{1, \dots, n\}} (-1)^{|I|+1} \mu(\bigwedge_{i \in I} \rho(\psi_i)) \right\} \\ &= \sup \{ \mu(\bigvee_{i=1}^n \rho(\psi_i)) \} \end{aligned} \quad (11.24)$$

by the inclusion-exclusion-formula of probability theory. The supremum carries over the same range as in (11.22). The family of elements $\bigvee_{i=1}^n \rho(\psi_i)$ in this supremum forms an upwards directed set in \mathcal{B} . By Lemma 9.1 we obtain therefore

$$\begin{aligned} f(\psi) &= \mu(\bigvee \{ \bigvee_{i=1}^n \rho(\psi_i) : \psi_i \in \mathcal{E}, \psi_i \geq \phi, i = 1, \dots, n; n = 1, 2, \dots \}) \\ &= \mu(\bigvee \{ \rho(\psi) : \psi \in \mathcal{E}, \psi \geq \phi \}) \\ &= \mu(\bar{\rho}(\psi)) \end{aligned}$$

Here, the associate law for joins in a complete lattice is used. Since $\bar{\rho}$ is an a.o.p, $f = \mu \circ \bar{\rho}$ is a support function on Ψ (see Theorem 11.5). This concludes the proof. \square

In the proof above we used the a.o.p ρ associated with the support function sp on \mathcal{E} . We remind that $sp_\nu = \mu \circ \rho = \mu \circ \rho_0 \circ s_\nu$. On the other hand the a.o.p $\bar{\rho}$ generates f , that is $f = \mu \circ \bar{\rho}$. From $sp_\nu = f$, as stated in the theorem, we deduce as a complement that $\rho = \rho_0 \circ s_\nu = \bar{\rho}$. In fact, we have seen that for $\psi \in \mathcal{E}$ we have $\rho(\psi) = \bar{\rho}(\psi)$ and for any $\phi \in \Psi$, $\phi \leq \psi \in \mathcal{E}$ implies $\rho(\psi) \leq \rho(\phi)$, hence $\bar{\rho}(\phi) \leq \rho(\phi)$. Then we have $\rho(\phi) = \bar{\rho}(\phi) \vee (\rho(\phi) - \bar{\rho}(\phi))$. It follows that

$$sp_\nu(\phi) = \mu(\rho(\phi)) = \mu(\bar{\rho}(\phi)) + \mu((\rho(\phi) - \bar{\rho}(\phi)))$$

But from $sp_\nu(\phi) = f(\phi) = \mu(\bar{\rho}(\phi))$ we deduce that $\mu(\rho(\phi) - \bar{\rho}(\phi)) = 0$, hence $\rho(\phi) - \bar{\rho}(\phi) = \perp$. Since $\bar{\rho}(\phi) \leq \rho(\phi)$ this means that indeed $\bar{\rho}(\phi) = \rho(\phi)$. We may rephrase this result in the following Corollary.

Corollary 11.3 *If $\rho = \rho_0 \circ s_\nu$ is the allocation of probability associated with the support function $sp_\nu = \mu \circ \rho$, which is the least extension of the support function sp on \mathcal{E} , then*

$$\rho(\phi) = \vee \{\rho(\psi) : \psi \in \mathcal{E}, \phi \leq \psi\}.$$

If the support function sp is defined on a lattice \mathcal{E} , then Theorem 11.15 may be sharpened (Shafer, 1973).

Theorem 11.16 *If sp is a support function defined on a lattice $\mathcal{E} \subseteq \Psi$, then*

$$sp_\nu(\phi) = \sup\{sp(\psi) : \psi \in \mathcal{E}, \phi \leq \psi\}. \quad (11.25)$$

Proof. Since sp_ν is monotone, the right hand side of (11.25) is less or equal to sp_ν . It remains to show the converse inequality. Again, let $\rho = \rho_0 \circ s_\nu$ be the a.o.p associated with the support function sp and μ the probability in the corresponding probability algebra (\mathcal{B}, μ) . Consider $\psi_1, \dots, \psi_n \in \mathcal{E}$. Since \mathcal{E} is a lattice, $\bigwedge_{i=1}^n \psi_i$ belongs to \mathcal{E} too. Note that

$$\begin{aligned} s_\nu(\bigwedge_{i=1}^n \psi_i) &= \{I \in I_\Psi : \bigwedge_{i=1}^n \psi_i \in \nu(I)\} \\ &\supseteq \bigcup_{i=1}^n \{I \in I_\Psi : \psi_i \in \nu(I)\} = \bigcup_{i=1}^n s_\nu(\psi_i). \end{aligned}$$

Therefore,

$$\begin{aligned} \rho(\bigwedge_{i=1}^n \psi_i) &= [s_\nu(\bigwedge_{i=1}^n \psi_i)] \geq [\bigcup_{i=1}^n s_\nu(\psi_i)] \\ &= \bigvee_{i=1}^n [s_\nu(\psi_i)] = \bigvee_{i=1}^n \rho(\psi_i). \end{aligned}$$

Here $[A]$ denotes, as usual, the projection of $A \in \mathcal{A}_\mathcal{E}$ to the associated Boolean algebra \mathcal{B} in the probability algebra (\mathcal{B}, μ) , see Section 9.2. Using (11.24) in the proof of Theorem 11.15 and $sp_\nu(\phi) = f(\phi)$, we obtain now

$$sp_\nu(\phi) = \sup\{\mu(\bigvee_{i=1}^n \rho(\psi_i))\} \leq \sup\{\mu(\rho(\bigwedge_{i=1}^n \psi_i))\}, \quad (11.26)$$

where the supremum ranges over $\psi_i \in \mathcal{E}$, $\phi \leq \psi_i$, $i = 1, \dots, n$ and $n = 1, 2, \dots$. From this it follows that

$$sp_\nu(\phi) = \sup\{\mu(\rho(\psi)) : \psi \in \mathcal{E}, \phi \leq \psi\} = \sup\{sp(\psi) : \psi \in \mathcal{E}, \phi \leq \psi\}.$$

This concludes the proof. \square

There are in particular several examples of compact information algebras where the finite elements form a lattice, hence where Theorem 11.16 applies if $\mathcal{E} = \Psi_f$. We give here a few examples

We have seen in Section 11.3, that support functions sp , defined on the finite elements Ψ_f of an algebraic information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ may

be extended either to a continuous support function sp_σ or to a condensable one sp_γ . Further, by definition of condensability, sp_γ is determined by the values of sp on Ψ_f . This is like sp_ν , which according to Theorem 11.15 is also determined by the values of sp on Ψ_f , if $\mathcal{E} = \Psi_f$. Does a similar result also hold for the continuous extension sp_σ ? Yes, but so far as we know, only for a very special case, namely if \mathcal{E} is a *distributive lattice*, see (Shafer, 1979), Theorem 4. The following theorem is a particular case of Shafer's result, a case of special interest for us, where we assume that the finite elements form a distributive lattice, like the cofinite elements in a subset algebra.

Theorem 11.17 *Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be an algebraic information algebra, $(\Psi_f, D; \leq, \perp, \cdot, \epsilon)$ a subalgebra of $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ and (Ψ_f, \leq) a distributive lattice. If sp is a support function defined on Ψ_f , then for all $\phi \in \Psi$,*

$$sp_\sigma(\phi) = \sup\{\lim_{i \rightarrow \infty} sp(\psi_i) : \psi_1 \leq \psi_2 \leq \dots \in \Psi_f, \bigvee_{i=1}^{\infty} \psi_i \geq \phi\}. \quad (11.27)$$

Proof. We denote the right hand side of (11.27) by f . Note that $\lim_{i \rightarrow \infty} sp(\psi_i) = sp_\sigma(\bigvee_{i=1}^{\infty} \psi_i) \leq sp_\sigma(\phi)$ if $\bigvee_{i=1}^{\infty} \psi_i \geq \phi$, see Theorem 11.4. This shows that $sp_\sigma \geq f$. We are going to show that f is a continuous support function extending sp . Since sp_σ is the minimal continuous support function extending sp (Theorem 11.12), this proves then that $sp_\sigma = f$.

Let (μ, \mathcal{B}) be the probability algebra associated with the canonical probability space of the support function sp and ρ the corresponding allocation of probability, so that $sp = \mu \circ \rho$. For each $\phi \in \Psi$ define $\mathcal{D}(\phi) \subseteq \mathcal{B}$ by

$$\mathcal{D}(\phi) = \{\bigwedge_{i=1}^{\infty} \rho(\psi_i) : \psi_i \in \Psi_f, i = 1, 2, \dots, \bigvee_{i=1}^{\infty} \psi_i \geq \phi\}$$

(here we follow the proof of Theorem 4 in (Shafer, 1979)). The sets $\mathcal{D}(\phi)$ are upwards directed: In fact, consider two countable sets $\psi_{1,i}, \psi_{2,j} \in \Psi_f$ such that $\bigvee_{i=1}^{\infty} \psi_{1,i}, \bigvee_{j=1}^{\infty} \psi_{2,j} \geq \phi$. Then, since Ψ_f is a lattice, the set $\psi_{1,i} \wedge \psi_{2,j}$ is still a countable subset of Ψ_f . And, since the lattice Ψ_f is distributive,

$$\bigvee_{i,j=1}^{\infty} (\psi_{1,i} \wedge \psi_{2,j}) = (\bigvee_{i=1}^{\infty} \psi_{1,i}) \wedge (\bigvee_{j=1}^{\infty} \psi_{2,j}) \geq \phi.$$

Finally, $\psi_{1,i} \wedge \psi_{2,j} \leq \psi_{1,i}, \psi_{2,j}$ implies $\rho(\psi_{1,i} \wedge \psi_{2,j}) \geq \rho(\psi_{1,i}), \rho(\psi_{2,j})$. So indeed, $\mathcal{D}(\phi)$ is upwards directed.

Define now $\tilde{\rho}(\phi) = \bigvee \mathcal{D}(\phi)$. We claim that $\tilde{\rho}$ is a σ -a.o.p and that $f = \mu \circ \tilde{\rho}$. This shows then that f is a continuous support function. Since obviously $f|_{\Psi_f} = sp$ this proves the theorem.

It is evident that $\tilde{\rho}(1) = \top$. So, it only remains to show that $\tilde{\rho}(\bigvee_{i=1}^{\infty} \psi_i) = \bigwedge_{i=1}^{\infty} \tilde{\rho}(\psi_i)$ or $\bigvee \mathcal{D}(\bigvee_{i=1}^{\infty} \psi_i) = \bigwedge_{i=1}^{\infty} \bigvee \mathcal{D}(\psi_i)$. Fix a sequence ψ_1, ψ_2, \dots . To

simplify notation let $\mathcal{D} = \mathcal{D}(\bigvee_{i=1}^{\infty} \psi_i)$, $\mathcal{D}_i = \mathcal{D}(\psi_i)$ and $M = \bigwedge_i \bigvee \mathcal{D}_i$. The task is then to show that

$$\bigvee \mathcal{D} = M.$$

Now, $\mathcal{D} \subseteq \mathcal{D}_i$ for all i , hence $\bigvee \mathcal{D} \leq \bigwedge_i \bigvee \mathcal{D}_i$. Further, since \mathcal{D}_i are upwards directed sets, by Lemma 9.1 we have

$$\mu(\bigvee \mathcal{D}_i) = \sup_{\psi \in \mathcal{D}_i} \mu(\psi).$$

Choose an $\epsilon > 0$. Then for all i , there is an $M_i \in \mathcal{D}_i$ such that

$$\mu(\bigvee \mathcal{D}_i - M_i) \leq \frac{\epsilon}{2i}.$$

Since $M \leq \bigvee \mathcal{D}_i$, we obtain also

$$\mu(M - M_i) = \mu(M \wedge M_i^c) \leq \mu(\bigvee \mathcal{D}_i \wedge M_i^c) = \mu(\bigvee \mathcal{D}_i - M_i) \leq \frac{\epsilon}{2i}.$$

Let B_i denote a set of elements $\phi_1, \phi_2, \dots \in \Psi_f$ such that $\bigvee B_i \geq \psi_i$ and $M_i = \bigwedge_{\phi \in B_i} \rho(\phi)$. Let $B_\epsilon = \bigcup_{i=1}^{\infty} B_i$ and $M_\epsilon = \bigwedge_{i=1}^{\infty} M_i$. Then $\bigvee B_\epsilon \geq \bigvee_{i=1}^{\infty} \psi_i$ and

$$M_\epsilon = \bigwedge_{i=1}^{\infty} \bigwedge_{\phi \in B_i} \rho(\phi) = \bigwedge_{\phi \in B_\epsilon} \rho(\phi).$$

Thus M_ϵ belongs to \mathcal{D} , hence $M_\epsilon \leq \bigvee \mathcal{D}$. We have

$$\begin{aligned} M - M_\epsilon &= M \wedge M_\epsilon^c = M \wedge \left(\bigwedge_{i=1}^{\infty} M_i \right)^c = M \wedge \left(\bigvee_{i=1}^{\infty} M_i^c \right) \\ &= \bigvee_{i=1}^{\infty} (M \wedge M_i^c) = \bigvee_{i=1}^{\infty} (M - M_i). \end{aligned}$$

Thus we obtain

$$\mu(M - M_\epsilon) = \mu\left(\bigvee_{i=1}^{\infty} (M - M_i)\right) \leq \epsilon.$$

Now, $M_\epsilon \leq \bigvee \mathcal{D}$ implies $M_\epsilon^c \geq (\bigvee \mathcal{D})^c$ and therefore $M - \bigvee \mathcal{D} = M \wedge (\bigvee \mathcal{D})^c \leq M \wedge M_\epsilon^c = M - M_\epsilon$. This shows that

$$\mu(M - \mathcal{D}) \leq \mu(M - M_\epsilon) \leq \epsilon$$

Since ϵ is arbitrarily small, we conclude that $\mu(M - \mathcal{D}) = 0$ and from this it follows that $\bigvee \mathcal{D} = M$, because $\bigvee \mathcal{D} \leq M$. This proves that $\tilde{\rho}$ is a σ -a.o.p.

Next, we are going to show that $f = \mu \circ \tilde{\rho}$, hence that f is a continuous support function. Note that $sp_\sigma = \mu \circ \rho$. Then, since $sp_\sigma|_{\Psi_f} = sp$ and since ρ is a σ -a.o.p, because sp_ν is continuous, we have

$$\begin{aligned} f(\phi) &= \sup\{\mu(\rho(\bigvee_{i=1}^{\infty} \psi_i)) : \psi_i \in \Psi_f, \bigvee_{i=1}^{\infty} \psi_i \geq \phi\} \\ &= \sup\{\mu(\rho(\chi)) : \chi \in \mathcal{D}(\phi)\}. \end{aligned}$$

Since $\mathcal{D}(\phi)$ is upwards directed, we obtain (Lemma 9.1)

$$f(\phi) = \mu(\bigvee \mathcal{D}(\phi)) = \mu(\tilde{\rho}(\phi))$$

This concludes the proof. \square

Under the assumptions of Theorem 11.17 we may, according to the considerations in the proof, also write

$$sp_\sigma(\phi) = \sup\{sp_\sigma(\bigvee_{i=1}^{\infty} \psi_i) : \psi_i \in \Psi_f, \bigvee_{i=1}^{\infty} \psi_i \geq \phi\},$$

or, equivalently,

$$sp_\sigma(\phi) = \sup\{sp_\sigma(\chi_i) : \chi_i \in \sigma(\Psi_f), \chi_i \geq \phi\},$$

If Ψ_f is in addition *countable*, then $\sigma(\Psi_f) = \Psi$ and

$$sp_\sigma(\psi) = \lim_{i \rightarrow \infty} sp(\psi_i)$$

if $\psi_1 \leq \psi_2 \leq \dots \in \Psi_f$ and $\bigvee_{i=1}^{\infty} \psi_i = \psi$. We remark that this result holds in general if the set of finite elements is countable, without the additional assumption that (Ψ_f, \leq) is a distributive lattice. This follows from the alternative approach to generate continuous support function, based on results of (Norberg, 1989) mentioned at the end of Section 11.2.

Just as Corollary 11.3, we may also derive the following result:

Corollary 11.4 *Under the conditions of Theorem 11.17, if ρ_σ is the a.o.p associated with the support function sp_σ , then $\rho_\sigma = \tilde{\rho}$, where the latter a.o.p is defined in the proof of Theorem 11.17.*

We have shown that a support function defined on some join-semilattice $\mathcal{E} \subseteq \Psi$ of an information algebra Ψ can have different kinds of extension, defined in terms of its values in \mathcal{E} . Similar and more results of this kind can be found in (Shafer, 1979) in a more restricted context.

11.5 The Boolean Case

In this section, the information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is assumed to be *Boolean*, that is, the semilattice Ψ is in fact a Boolean algebra. Everything said so far about random mappings, allocations of probability and support functions remains valid. However the Boolean nature of Ψ allows to present an equivalent dual view to allocations of probability and support functions. This dual view comes from considering possibility sets and associated degrees of plausibility as introduced in Chapter 9. In a general information algebra these concepts are of no particular interest, they are far less interesting and important than allocations of support and support functions. In the Boolean case however their status changes to one of equal importance and interest.

Consider a random mapping Γ from a probability space (Ω, \mathcal{A}, P) to a Boolean information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$. Generalising the discussion in Section 9.1 with respect to simple random variables, we define the set of assumptions ω under which a hypothesis $\psi \in \Psi$ is *possible*, that is not excluded, by

$$p_\Gamma(\psi) = \{\omega \in \Omega : \psi \cdot \Gamma(\omega) \neq 0\}.$$

Given, that the top element 0 of the Boolean algebra (Ψ, \leq) is considered to represent the contradiction, an assumption ω such that $\psi \cdot \Gamma(\omega) = 0$ must be considered as impossible, as excluded by the information contained in the random mapping Γ . Therefore, $p_\Gamma(\psi)$ is called the *possibility set* of ψ , relative to the random mapping Γ .

In a Boolean algebra we have $\psi \cdot \Gamma(\omega) = \psi \vee \Gamma(\omega) = 0$ if and only if $\psi^c \leq \Gamma(\omega)$, where ψ^c denotes the complement of ψ in Ψ . Therefore, we see that

$$p_\Gamma(\psi) = \{\omega \in \Omega : \psi^c \leq \Gamma(\omega)\}^c = (s_\Gamma(\psi^c))^c, \quad (11.28)$$

where s_Γ is the allocation of support associated with the random mapping Γ (see (9.5)). This is the first of the basic duality relations between support and plausibility or possibility considered in this section. It allows to translate results on allocations of support immediately to possibility sets.

Theorem 11.18 *If $\Gamma : \Omega \rightarrow \Psi$, where $(\Psi; \leq)$ is a Boolean algebra, then*

1. $p_\Gamma(0) = \emptyset$,
2. *If $\phi \leq \psi$, then $p_\Gamma(\psi) \subseteq p_\Gamma(\phi)$.*
3. $p_\Gamma(\phi \wedge \psi) = p_\Gamma(\phi) \cup p_\Gamma(\psi)$.
4. *If Γ is normalised, then $p_\Gamma(1) = \Omega$.*

5. If Ψ is a Boolean σ -algebra, then

$$p_\Gamma(\bigwedge_{i=1}^{\infty} \psi_i) = \bigcup_{i=1}^{\infty} p_\Gamma(\psi_i).$$

6. If Ψ is a complete Boolean algebra, then for any subset X of Ψ ,

$$p_\Gamma(\bigwedge X) = \bigcup_{\psi \in X} p_\Gamma(\psi).$$

Proof. Items (1) to (4) follow immediately from Theorem 11.1 and the duality relation (11.28). Items (5) and (6) follow similarly from Theorem 11.2, (11.28) and de Morgan laws. \square

If $p_\Gamma(\psi)$ is measurable, the probability, that ψ is not excluded by Γ , $pl_\Gamma(\psi) = P(p_\Gamma(\psi))$ is defined. This is called the *degree of possibility* or *plausibility* of ψ under the random mapping Γ . Let $\mathcal{Z}_\Gamma = \{\psi \in \Psi : p_\Gamma(\psi) \in \mathcal{A}\}$ be the set of ψ for which $p_\Gamma(\psi)$ is measurable. Recall that \mathcal{E}_Γ is the set of elements of Ψ for which $s_\Gamma(\psi)$ is measurable. Clearly, $\psi \in \mathcal{Z}_\Gamma$ implies $\psi^c \in \mathcal{E}_\Gamma$. According to Theorem 11.3, \mathcal{E}_Γ is a join-semilattice, containing 1. Thus \mathcal{Z}_Γ is a meet-semilattice, containing 0. Let's fix this result in the following theorem.

Theorem 11.19 *If $(\Psi; \leq)$ is a Boolean algebra, Γ a random mapping into Ψ , then \mathcal{Z}_Γ is a meet-subsemilattice of Ψ containing 0. If Γ is normalised, then 1 belongs to \mathcal{Z}_Γ too. If $(\Psi; \leq)$ is a Boolean σ -algebra, then \mathcal{Z}_Γ is a σ -semilattice.*

Note that

$$pl_\Gamma(\psi) = P(p_\Gamma(\psi)) = P((s_\Gamma(\psi^c))^c) = 1 - sp_\Gamma(\psi^c). \quad (11.29)$$

This is a second duality relation between support and plausibility in a Boolean algebra.

The function $pl_\Gamma : \mathcal{Z}_\Gamma \rightarrow [0, 1]$ is called the *plausibility function* associated with the random mapping Γ . Just as the support function sp_Γ can be extended from \mathcal{E}_Γ to Ψ by defining $sp_\Gamma = \mu \circ \rho_\Gamma$, where (μ, \mathcal{B}) is the probability algebra associated to the probability space (Ω, \mathcal{A}, P) and $\rho_\Gamma = \rho_0 \circ s_\Gamma$ the allocation of probability associated with Γ , we may extend pl_Γ in a similar way, see Section 9.2. This is done with the help of ξ_0 as defined by

$$\xi_0(H) = (\rho_0(H^c))^c = \bigwedge \{[A] : A \supseteq H, A \in \mathcal{A}\},$$

and $\xi_\Gamma = \xi_0 \circ p_\Gamma$ and $pl_\Gamma = \mu \circ \xi_\Gamma$ (see Section 9.2). Then we obtain

$$\begin{aligned} pl_\Gamma(\psi) &= \mu(\xi_\Gamma(\psi)) = \mu(\xi_0(p_\Gamma(\psi))) = \mu((\rho_0((p_\Gamma(\psi))^c))^c) \\ &= \mu((\rho_0(s_\Gamma(\psi^c)))^c) = \mu((\rho_\Gamma(\psi^c))^c) = 1 - sp_\Gamma(\psi^c). \end{aligned}$$

So, the extension $pl_\Gamma = \mu \circ \xi_\Gamma$ of the plausibility function to Ψ maintains the duality relation to the support function. Further, we have seen in Section 9.2 that $p_\Gamma(\psi) = P^*(p_\Gamma(\psi))$, where P^* is the outer probability measure of P .

In the present case of a Boolean algebra Ψ , we note that

$$\xi_\Gamma(\psi) = \xi_0(p_\Gamma(\psi)) = \xi_0((s_\Gamma(\psi^c))^c) = (\rho_0(s_\Gamma(\psi^c)))^c = (\rho_\Gamma(\psi^c))^c.$$

Here we have a third duality relation, which implies immediately, that $\xi(0) = \perp$ and $\xi(\psi \wedge \psi) = \xi(\psi) \vee \xi(\psi)$. A function from Ψ to \mathcal{B} with these two properties is called an *allowment of probability* (Shafer, 1979).

Definition 11.3 Allowment of probability. *If $(\Psi; \leq)$ is a Boolean algebra, (μ, \mathcal{B}) a probability algebra, then an allowance of probability is a mapping $\xi : \Psi \rightarrow \mathcal{B}$ such that*

1. $\xi(0) = \perp$,
2. $\xi(\psi \wedge \psi) = \xi(\psi) \vee \xi(\psi)$.

If furthermore, $\xi(1) = \top$ holds, then the allowance is called normalised.

To any allocation of probability $\rho : \Psi \rightarrow \mathcal{B}$ we associate an allowance of probability $\xi : \Psi \rightarrow \mathcal{B}$ defined by

$$\xi(\psi) = (\rho(\psi^c))^c \quad (11.30)$$

and vice versa to any allowance of probability ξ , an allocation of probability ρ , defined by $\rho(\psi) = (\xi(\psi^c))^c$ is associated.

In order to exploit this duality we consider the *dual Boolean algebra* $(\Psi^{op}; \leq_{op})$ of (Ψ, \leq) , with inverse order \leq_{op} and the corresponding dual meet \wedge_{op} and join \vee_{op} , so that

$$\begin{aligned} \psi \leq_{op} \phi & \quad \text{if and only if} \quad \phi \leq \psi, \\ \phi \vee_{op} \psi & \quad = \quad \phi \wedge \psi = (\phi^c \vee \psi^c)^c, \\ \phi \wedge_{op} \psi & \quad = \quad \phi \vee \psi = (\phi^c \wedge \psi^c)^c, \\ 0_{op} & \quad = \quad 1, \end{aligned}$$

To any extraction operator ϵ_x for $x \in D$ we associate a mapping $\epsilon_x^{op} : \Psi^{op} \rightarrow \Psi^{op}$ defined by

$$\epsilon_x^{op}(\psi) = (\epsilon_x(\psi^c))^c.$$

If we interpret dual join \vee_{op} as (dual) combination \cdot_{op} and the maps ϵ_x^{op} as (dual) extraction, then it turns out, that $(\Psi^{op}, D; \leq, \perp, \cdot_{op}, \epsilon^{op})$ is in fact still a Boolean information algebra. To verify this claim, we note first that $\epsilon_x(\psi) = \psi$ implies $\epsilon_x^{op}(\psi) = \psi$, that is, if x is a support of ψ with

respect to ϵ , it is also a support of ψ with respect to ϵ^{op} : $\epsilon_x(\psi) = \psi$ implies $0 = \epsilon_x(0) = \epsilon_x(\psi \cdot \psi^c) = \psi \cdot \epsilon_x(\psi^c)$, hence $\psi^c \leq \epsilon_x(\psi^c)$ and therefore $\psi^c = \epsilon_x(\psi^c)$. But then, $\epsilon_x^{op}(\psi) = (\psi^c)^c = \psi$. Assume further $x \perp y | z$ and that x is a support of ψ . Then, by Axiom A4 in the original Boolean information algebra, we have

$$\epsilon_y^{op}(\psi) = (\epsilon_y(\psi^c))^c = (\epsilon_y(\epsilon_z(\psi^c)))^c = (\epsilon_y(\epsilon_z^{op}(\psi)^c))^c = \epsilon_y^{op}(\epsilon_z^{op}(\psi)).$$

So, Axiom A4 holds also with respect to ϵ_x^{op} . Finally suppose that x is a support of ϕ and y a support of ψ and still $x \perp y | z$. Then by Axiom A5, we have

$$\begin{aligned} \epsilon_z(\phi \cdot_{op} \psi) &= (\epsilon_z((\phi \cdot_{op} \psi)^c))^c = (\epsilon_z(\phi^c \cdot \psi^c))^c \\ &= (\epsilon_z(\phi^c) \cdot \epsilon_z(\psi^c))^c = (\epsilon_z(\phi^c))^c \cdot_{op} (\epsilon_z(\psi^c))^c = \epsilon_z^{op}(\phi) \cdot_{op} \epsilon_z^{op}(\psi). \end{aligned}$$

Axiom A5 is also valid in the dual setting. This shows that $(\Psi^{op}, D; \leq, \perp, \cdot_{op}, \epsilon^{op})$ is indeed a Boolean information algebra.

For later reference let's also consider the dual of an *algebraic* Boolean information algebra with finite elements Ψ_f . Then (Ψ, \leq) is a complete lattice and, therefore, (Ψ, \leq_{op}) is a complete lattice too. Define the set

$$\Psi_{cf} = \{\psi : \psi^c \in \Psi_f\} \quad (11.31)$$

whose elements are called *cofinite*. Density in Ψ leads by de Morgan laws to

$$\phi = \bigvee_{op} \{\psi \in \Psi_{cf} : \psi \leq_{op} \phi\}.$$

Similary, strong density implies

$$\epsilon_x^{op}(\phi) = \bigvee_{op} \{\psi \in \Psi_{cf} : \psi = \epsilon_x^{op}(\psi) \leq_{op} \phi\}.$$

Thus, the dual information algebra $(\Psi^{op}, D; \leq, \perp, \cdot_{op}, \epsilon_x^{op})$ is also *algebraic* and the cofinite elements of (Ψ, \leq) are its finite elements..

As an example consider multivariate algebras.

Example 11.1 Dual Multivariate Algebras: Let $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ be a multivariate set algebra (see Section 5.2) in a set Ω_I , where I is an index set and

$$\Omega_I = \prod_{i \in I} \Omega_i$$

and Ω_i are sets of possible values for variables X_i , $i \in I$. Elements of $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ are subsets of Ω_I . This is a Boolean information algebra where join is intersection, meet is union. The (finite) subsets s of I form

the lattice D and extraction relative to $s \in D$ is defined as s -saturation, that is as saturation relative to the partition of Ω_I induced by the subset s of the index set I . In the dual information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ join is union, meet intersection. Dual extraction is defined according to (11.31) by $\sigma_s^{op}(S) = (\sigma_s(S^c))^c$, for any subset S of Ω_I .

The algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is *algebraic*, its finite elements are the *cofinite* sets of Ω_I , that is the complements of finite subsets of Ω_I . The cofinite elements of Ψ , that is the finite elements of Ψ^{op} , are the *finite* subsets of Ω_I . \ominus

Now we have the means to exploit duality between allocations and allowments of probability (11.30) and between degrees of support and plausibility (11.29). Let $\rho : \Psi \rightarrow \mathcal{B}$ be an allocation of probability to a Boolean information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ relative to a probability algebra (μ, \mathcal{B}) . The corresponding allowment of probability ξ , defined by (11.30) can be seen as a mapping $\xi : \Psi^{op} \rightarrow \mathcal{B}^{op}$ between the dual Boolean algebras of Ψ and \mathcal{B} . Then, in this view, ξ is an *allocation* of probability, that is

1. $\xi(\mathbf{o}_{op}) = \top_{op}$,
2. $\xi(\phi \vee_{op} \psi) = \xi(\phi) \wedge_{op} \xi(\psi)$.

As a consequence, as allocations of probability, the ξ form an idempotent generalised information algebra $(A_{\Psi^{op}}, D; \leq, \perp, \cdot, \epsilon^{op})$ (see Theorem 10.6). Let's denote combination by \vee_{op} , such that according to (10.2)

$$\begin{aligned} (\xi_1 \vee_{op} \xi_2)(\psi) &= \bigvee_{op} \{\xi_1(\psi_1 \wedge_{op} \xi_2(\psi_2)) : \psi \leq_{op} \psi_1 \vee_{op} \psi_2\} \\ &= \bigwedge \{\xi_1(\psi_1 \vee \xi_2(\psi_2)) : \psi \geq \psi_1 \wedge \psi_2\}. \end{aligned}$$

Similarly, for extraction, we obtain, using (10.5),

$$\begin{aligned} \epsilon_x^{op}(\xi)(\phi) &= \bigvee_{op} \{\xi(\psi) : \psi = \epsilon_x^{op}(\psi) \geq_{op} \phi\} \\ &= \bigwedge \{\xi(\psi) : \psi = x(\psi) \leq \phi\}. \end{aligned}$$

Clearly, by the map $\rho \mapsto \xi$, defined by $\xi(\psi) = \rho(\psi^c)^c$, is an isomorphism between information algebras.

We write $\xi_1 \leq_{op} \xi_2$ if $\xi_1 \vee_{op} \xi_2 = \xi_2$. Then, $\xi_1 \leq_{op} \xi_2$ if and only if $\xi_1(\psi) \leq_{op} \xi_2(\psi)$ for all $\psi \in \Psi^{op}$. If we look at this relative to the original algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$, then it is convenient to write $\xi_1 \wedge \xi_2 = \xi_1 \vee_{op} \xi_2$ and hence $\xi_1 \geq \xi_2$ if $\xi_1 \leq_{op} \xi_2$. Finally, we write simply $\epsilon_x(\xi)$ instead of $\epsilon_x^{op}(\xi)$ for $x \in D$. In the following we shall use this convention.

Next, we use the duality relation (11.30) to translate results relating random mappings to allocations of probability obtained in Section 10.2 to allowments of probability. Here is a list of such results, which can be easily obtained by (11.30) and de Morgan laws:

1. If Δ_1, Δ_2 and Δ are simple random variables, then by (10.12)

$$\begin{aligned}\xi_{\Delta_1 \cdot \Delta_2} &= \xi_{\Delta_1} \wedge \xi_{\Delta_2}, \\ \xi_{\epsilon_x(\Delta)} &= \epsilon_x(\xi_\Delta).\end{aligned}$$

2. If Γ is a generalised random variable, then (Theorem 10.8)

$$\xi_\Gamma = \bigwedge \{\xi_\Delta : \Delta \leq \Gamma\}.$$

Here, Δ denote as usual simple random variables.

3. if Γ_1, Γ_2 and Γ are generalised random variables, then (Theorem 10.9)

$$\begin{aligned}\xi_{\Gamma_1 \cdot \Gamma_2} &= \xi_{\Gamma_1} \wedge \xi_{\Gamma_2}, \\ \xi_{\epsilon_x(\Gamma)} &= \epsilon_x(\xi_\Gamma).\end{aligned}\tag{11.32}$$

4. If Γ is a generalised random variable, $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ an algebraic Boolean information algebra, $X \subseteq \Psi$ a downwards directed set, then (Theorem 10.10)

$$\xi_\Gamma(\bigwedge X) = \bigvee_{\psi \in X} \xi_\Gamma(\psi).$$

5. Suppose $(\Psi, D, \leq, \perp, \cdot, \epsilon)$ is an algebraic information algebra and $\Gamma_i \in \mathcal{R}_\sigma$ for $i = 1, 2, \dots$, then (Theorem 10.11)

$$\xi_{\bigvee_{i=1}^{\infty} \Gamma_i} = \bigwedge_{i=1}^{\infty} \xi_{\Gamma_i}.$$

6. If Γ_i form a montone sequence random variables $\Gamma_1 \leq \Gamma_2 \leq \dots$, $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ in an algebraic Boolean information algebra, then (Theorem 10.12)

$$\epsilon_x(\bigwedge_{i=1}^{\infty} \xi_{\Gamma_i}) = \bigwedge_{i=1}^{\infty} \epsilon_x(\xi_{\Gamma_i}).$$

Now we turn to plausibility and exploit duality relation (11.29) to derive results on degrees of plausibility from support functions. If Γ is a random map, mapping a probability space into an idempotent generalised information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ (or its ideal completion), then recall that its support function is defined by $sp_\Gamma = \mu \circ \rho_\Gamma$, where $\rho_\Gamma = \rho_0 \circ s_\Gamma$ and (\mathcal{B}, μ) is the probability algebra associated with the probability space (see Section 9.2). Similarly, the associated degrees of plausibility pl_Γ , related to sp_Γ by the duality relation (11.29), is given by $pl_\Gamma = \mu \circ \xi_\Gamma$, where $\xi_\Gamma = \xi_0 \circ p_\Gamma$. And ρ_Γ and ξ_Γ are related by the duality relation (11.30).

Here follows a list of results on plausibility, derived from corresponding results on support function via the duality relation (11.29):

1. Let Γ be a random mapping, then (Theorem 11.4)

(a) $pl_{\Gamma}(0) = 0$.

(b) If $\psi_1, \dots, \psi_n \leq \psi$, $\psi_1, \dots, \psi_m, \psi \in \mathcal{Z}_{\Gamma}$,

$$pl_{\Gamma}(\psi) \leq \sum_{\emptyset \neq I \subseteq \{1, \dots, m\}} (-1)^{|I|+1} pl_{\Gamma}(\bigwedge_{i \in I} \psi_i). \quad (11.33)$$

(c) If \mathcal{Z}_{Γ} is a σ -meet semilattice, and if $\psi_1 \geq \psi_2 \geq \dots \in \mathcal{Z}_{\Gamma}$, then

$$pl_{\Gamma}(\bigwedge_{i=1}^{\infty} \psi_i) = \lim_{i \rightarrow \infty} pl_{\Gamma}(\psi_i). \quad (11.34)$$

(d) If Γ is normalised, then $pl_{\Gamma}(1) = 1$.

2. If (\mathcal{B}, μ) is a probability algebra and $\xi : \Psi \rightarrow \mathcal{B}$ is an allotment of probability and $pl = \mu \circ \xi$, then (Theorem 11.5)

(a) pl satisfies properties (a) and (b) of item 1 above.

(b) If Ψ is a σ -meet-semilattice and if for all ψ_1, ψ_2, \dots , we have $\xi(\bigwedge_{i=1}^{\infty} \psi_i) = \bigvee_{i=1}^{\infty} \xi(\psi_i)$, then (c) of item 1 above holds.

(c) If Ψ is a complete lattice and if for any downwards directed set $X \subseteq \Psi$

$$\xi(\bigwedge X) = \bigvee_{\psi \in X} \xi(\psi)$$

holds, then

$$pl(\bigwedge X) = \sup_{\psi \in X} pl(\psi). \quad (11.35)$$

3. If Γ is a generalised random variable, $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ an algebraic Boolean information algebra, $pl_{\Gamma} = \mu \circ \xi_{\Gamma}$, $\xi_{\Gamma} = \xi_0 \circ p_{\Gamma}$, then (Theorem 11.6)

$$pl_{\Gamma}(\psi) = \sup\{pl_{\Gamma}(\phi) : \psi \in \Psi_{cf}, \phi \geq \psi\}.$$

Furthermore, if $X \subseteq \Psi$ is downwards directed, then

$$pl_{\Gamma}(\bigwedge X) = \sup_{\psi \in X} pl_{\Gamma}(\psi).$$

4. Let $(\sigma(\Psi), D; \leq, \perp, \cdot, \epsilon)$ the σ -extension of the Boolean information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$, Γ a random variable, that is, $\Gamma = \bigvee_{i=1}^{\infty} \Delta_i$, where Δ_i is a monotone increasing sequences of simple random variables with values in Ψ , then for all $\psi \in \Psi$ (Theorem 11.7)

$$pl_{\Gamma}(\psi) = \lim_{i \rightarrow \infty} pl_{\Delta_i}(\psi).$$

5. If Γ is a generalised random variable, then for all $\psi \in \Psi$ (Corollary 11.1)

$$pl_{\Gamma}(\psi) = \inf\{pl_{\Delta}(\psi) : \Delta \leq \Gamma\},$$

where Δ as usual are simple random variables.

These results allow to give a dual version of Definition 11.2, now regarding plausibility functions:

Definition 11.4 *Let \mathcal{Z} be a meet-semilattice with a top element 0. Then a function $pl : \mathcal{Z} \rightarrow [0, 1]$ satisfying (1) and (2) below is called a plausibility function on \mathcal{Z} :*

1. $pl(0) = 0$.
2. If $\psi_1, \dots, \psi_m \leq \psi$, $\psi_1, \dots, \psi_m, \psi \in \mathcal{Z}$,

$$pl(\psi) \leq \sum_{\emptyset \neq I \subseteq \{1, \dots, m\}} (-1)^{|I|+1} pl(\bigwedge_{i \in I} \psi_i). \quad (11.36)$$

3. If in addition \mathcal{Z} is closed under countable meets, and for any montone sequence $\psi_1 \geq \psi_2 \geq \dots$ the condition

$$pl\left(\bigwedge_{i=1}^{\infty} \psi_i\right) = \lim_{i \rightarrow \infty} pl(\psi_i) \quad (11.37)$$

holds, then pl is called a continuous plausibility function of \mathcal{Z} .

4. If further \mathcal{Z} is a complete meet-semilattice and for any downwards directed set $X \subseteq \mathcal{Z}$,

$$pl\left(\bigwedge X\right) = \sup_{\psi \in X} pl(\psi) \quad (11.38)$$

holds, then pl is called a condensable plausibility function on \mathcal{Z} .

A function satisfying (2) above is also called *alternating of order ∞* (Choquet, 1953–1954). Thus, the degrees of plausibility of any random mapping Γ form a plausibility function. If Γ is a generalised random variable in an algebraic Boolean information algebra, then pl_{Γ} is condensable, and if Γ is a random variable, then pl_{Γ} is continuous.

Given a plausibility function pl on a meet-semilattice $\mathcal{Z} \subseteq \Psi$, where $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is a Boolean information algebra, the function $sp(\psi) = 1 - pl(\psi^c)$ is a support function on a join-semilattice $\mathcal{E} \subseteq \Psi$. Based on this remark we conclude that there is a random mapping generating sp , hence pl . In fact, the canonical random mapping ν (see Section 11.3) generates the plausibility function $pl_{\nu}(\psi) = 1 - sp_{\nu}(\psi^c)$ on Ψ , which is the *maximal*

extension of pl from \mathcal{Z} to Ψ . If the Boolean information algebra $(\Psi, D; \leq, \perp, \cdot, \epsilon)$ is algebraic, the random mapping σ (11.16) generates the maximal *continuous* extension $pl_\sigma(\psi) = 1 - sp_\sigma(\psi^c)$ (see Theorem 11.12). And the random mapping γ (11.17) generates according to (11.19) a *condensable* plausibility function (Theorem 11.14). This concludes the duality discussion between support and plausibility in Boolean information algebras.

References

- Beeri, C., Fagin, R., Maier, D., & Yannakakis, M. 1983. On the Desirability of Acyclic Database Schemes. *Journal of the ACM*, **30**(3), 479–513.
- Billingsley, P. 1995. *Probability and Measure*. John Wiley, New York.
- Choquet, G. 1953–1954. Theory of Capacities. *Annales de l'Institut Fourier*, **5**, 131–295.
- Choquet, G. 1969. *Lectures on Analysis*. Benjaminm, New York.
- Clifford, A. H., & Preston, G. B. 1967. *Algebraic Theory of Semigroups*. Providence, Rhode Island: American Mathematical Society.
- Cowell, R. G., Dawid, A. P., Lauritzen, S. L., & Spiegelhalter, D. J. 1999. *Probabilistic Networks and Expert Systems*. Information Sci. and Stats. Springer, New York.
- Cuzzolin, F. 2005. Algebraic Structure of the Families of Compatible Frames of Discernment. *Ann. of Mathematics and Artificial Intelligence*, **45**, 241–274.
- Davey, B.A., & Priestley, H.A. 1990. *Introduction to Lattices and Order*. Cambridge University Press.
- Davey, B.A., & Priestley, H.A. 2002. *Introduction to Lattices and Order*. Cambridge University Press.
- Dawid, A. P. 2001. Separoids: A Mathematical Framework for Conditional Independence and Irrelevance. *Ann. Math. Artif. Intell.*, **32**(1–4), 335–372.
- Dechter, R. 1999. Bucket Elimination: A Unifying Framework for Reasoning. *Artificial Intelligence*, **113**, 41–85.
- Dempster, A.P. 1967a. Upper and Lower Probabilities Induced by a Multivalued Mapping. *Annals of Math. Stat.*, **38**, 325–339.
- Dempster, A.P. 1967b. Upper and Lower Probability Inferences Based on a Sample from a Finite Univariate Population. *Biometrika*, **54**, 515–528.

- Gierz, et. al. G. 2003. *Continuous Lattices and Domains*. Cambridge University Press.
- Gottlob, G., Leone, N., & Scarcello, F. 1999a. A Comparison of Structural CSP Decomposition Methods. *Pages 394–399 of: Proceedings of the 16th International Joint Conference on Artificial Intelligence IJCAI*. Morgan Kaufmann.
- Gottlob, G., Leone, N., & Scarcello, F. 1999b. Hypertree decompositions and tractable queries. *Pages 21–32 of: PODS '99: Proceedings of the eighteenth ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems*. New York, NY, USA: ACM Press.
- Gottlob, G., Leone, N., & Scarcello, F. 2001. The complexity of acyclic conjunctive queries. *J. ACM*, **48**(3), 431–498.
- Grätzer, G. 1978. *General Lattice Theory*. Academic Press.
- Guan, Xuechong, & Li, Yongming. 2010. On Compact Information Algebra. *Unpublished paper*, 1–16.
- Guan, Xuechong, & Li, Yongming. 2012. On Two Types of Continuous Information Algebras. *Int. J. Unc. Fuzzy Knowl. Based Syst.*, **20**, 655–672.
- Guang, X, Li Y., & Kohlas, J. 2015. On Conditions for Semirings to Induce Compact Information Algebras. *Mathematical Structures in Computer Science*, 1–10.
- Haenni, R., Kohlas, J., & Lehmann, N. 2000. Probabilistic Argumentation Systems. *Pages 221–287 of: Kohlas, J., & Moral, S. (eds), Handbook of Defeasible Reasoning and Uncertainty Management Systems, Volume 5: Algorithms for Uncertainty and Defeasible Reasoning*. Kluwer, Dordrecht.
- Halmos, Paul R. 1963. *Lectures on Boolean Algebras*. Van Nostrand-Reinhold.
- Jirousek, R. 1997. Composition of Probability Measures on Finite Spaces. *Pages 274–281 of: Geiger, D., & Shenoy, P. (eds), Uncertainty in Artificial Intelligence*. UAI. Morgan Kaufmann.
- Jirousek, R. 2011. Foundations of Compositional Model Theory. *Int. J. of General Systems*, **40**, 623–678.
- Jirousek, R., & Shenoy, P. 2014. Compositional Models in Valuation Based Systems. *Int. J. of Approximate Reasoning*, **55**, 277–293.

- Jirousek, R., & Shenoy, P. 2015. Causal Compositional Models in Valuation Based Systems with Examples in Specific Theories. *Int. J. of Approximate Reasoning*.
- Kappos, D. A. 1969. *Probability Algebras and Stochastic Spaces*. New York: Academic Press.
- Kelley, J.L. 1955. *General Topology*. D. Van Nostrand Company, Princeton, New Jersey.
- Kohlas, J. 1997. Allocation of Arguments and Evidence Theory. *Theoretical Computer Science*, **171**, 221–246.
- Kohlas, J. 2003a. *Information Algebras: Generic Structures for Inference*. Springer-Verlag.
- Kohlas, J. 2003b. Probabilistic Argumentation Systems. A New Way to Combine Logic with Probability. *J. of Applied Logic*, **1**, 225–253.
- Kohlas, J., & Monney, P.-A. 2007. An algebraic theory for statistical information based on the theory of hints. *Int. J. Approx. Reason.*, **doi:10.1016/j.ijar.2007.05.003**.
- Kohlas, J., & Monney, P.A. 1994. Advances in Dempster-Shafer Theory of Evidence. *Pages 473–492 of: R.R. Yager, J. Kacprzyk, & Fedrizzi, M. (eds), Representation of Evidence by Hints*. Wiley.
- Kohlas, J., & Monney, P.A. 1995. *A Mathematical Theory of Hints. An Approach to the Dempster-Shafer Theory of Evidence*. Lecture Notes in Economics and Mathematical Systems, vol. 425. Springer.
- Kohlas, J., & Schmid, J. 2014. An Algebraic Theory of Information: An Introduction and Survey. *Information*, **5**, 219–254.
- Kohlas, J., & Schmid, J. 2016. *Commutative Information Algebras and Their Representation Theory*. To be published, Department of Informatics, University of Fribourg.
- Kohlas, J., & Shenoy, P.P. 2000. Computation in Valuation Algebras. *Pages 5–39 of: Kohlas, J., & Moral, S. (eds), Handbook of Defeasible Reasoning and Uncertainty Management Systems, Volume 5: Algorithms for Uncertainty and Defeasible Reasoning*. Kluwer, Dordrecht.
- Kohlas, J., & Wilson, N. 2006. Exact and Approximate Local Computation in Semiring Induced Valuation Algebras. *Artificial Intelligence*, **172**, 1360–1399.

- Kohlas, Jürg, & Schneuwly, Cesar. 2009. Information Algebra. *Pages 95–127 of: Sommaruga, Giovanni (ed), Formal Theories of Information.* Lecture Notes in Computer Science, vol. 5363. Springer.
- Lauritzen, S. L., & Jensen, F. V. 1997. Local Computation with Valuations from a Commutative Semigroup. *Ann. Math. Artif. Intell.*, **21**(1), 51–69.
- Lauritzen, S. L., & Spiegelhalter, D. J. 1988. Local computations with probabilities on graphical structures and their application to expert systems. *J. Royal Statist. Soc. B*, **50**, 157–224.
- Maier, D. 1983. *The Theory of Relational Databases*. London: Pitman.
- Mitsch, H. 1986. A Natural partial Order for Semigroups. *Proc. Amer. Math. Soc.*, **97**, 384–388.
- Nambooripad, K.S.S. 1980. The Natural Partial Order of a Regular Semigroup. *Proc. Edinburgh Math. Soc.*, **23**, 249–260.
- Norberg, T. 1989. Existence theorems for measures on continuous posets, with applications to random set theory. *Math. Scand.*, **64**, 15–51.
- Phelps, R.R. 2001. *Lectures on Choquet's Theorem*. Springer, Lecture Notes in Mathematics.
- Pouly, M., & Kohlas, J. 2011. *Generic Inference. A Unified Theory for Automated Reasoning*. Wiley, Hoboken, New Jersey.
- Shafer, G. 1973. *Allocation of Probability: A Theory of Partial Belief*. Ph.D. thesis, Princeton University.
- Shafer, G. 1976. *A Mathematical Theory of Evidence*. Princeton University Press.
- Shafer, G. 1979. Allocations of Probability. *Ann. of Prob.*, **7**, 827–839.
- Shafer, G. 1991. *An Axiomatic Study of Computation in Hypertrees*. Working Paper 232. School of Business, University of Kansas.
- Shafer, G. 1996. *Probabilistic Expert Systems*. CBMS-NSF Regional Conference Series in Applied Mathematics, no. 67. Philadelphia, PA: SIAM.
- Shafer, G., & Shenoy, P. 1990. Axioms for Probability and Belief Function Propagation. In: Shafer, G., & Pearl, J. (eds), *Readings in Uncertain Reasoning*. Morgan Kaufmann Publishers Inc., San Mateo, California.
- Shafer, G., Shenoy, P.P., & Mellouli, K. 1987. Propagating Belief Functions in Qualitative Markov Trees. *Int. J. of Approximate Reasoning*, **1**(4), 349–400.

- Shenoy, P. P., & Shafer, G. 1990a. Axioms for probability and belief-function proagation. *Pages 169–198 of*: Shachter, Ross D., Levitt, Tod S., Kanal, Laveen N., & Lemmer, John F. (eds), *Uncertainty in Artificial Intelligence 4*. Machine intelligence and pattern recognition, vol. 9. Amsterdam: Elsevier.
- Shenoy, P.P. 1992. Valuation-Based Systems: A Framework for Managing Uncertainty in Expert Systems. *Pages 83–104 of*: Zadeh, L.A., & Kacprzyk, J. (eds), *Fuzzy Logic for the Management of Uncertainty*. John Wiley & Sons.
- Shenoy, P.P. 1994a. Conditional Independence in Valuation-based Systems. *International Journal of Approximate Reasoning*, **10**, 203–234.
- Shenoy, P.P. 1994b. Using Dempster-Shafer’s Belief Function Theory in Expert Systems. *Pages 395–414 of*: R.R. Yager, J. Kacprzyk, & Fedrizzi, M. (eds), *Advances in The Dempster-Shafer Theory of Evidence*. John Wiley & Sons.
- Shenoy, P.P. 1996. Axioms for Dynamic Programming. *Pages 259–275 of*: Gammernan, A. (ed), *Computational Learning and Probabilistic Reasoning*. Wiley, Chichester, UK.
- Shenoy, P.P., & Shafer, G. 1990b. Axioms for Probability and Belief Function Propagation. *Pages 169–198 of*: R.D. Shachter, T.S. Levitt, J.F. Lemmer, & Kanal, L.N. (eds), *Uncertainty in Artif. Intell. 4*. North Holland.
- Stoltenberg-Hansen, V., Lindstroem, I., & Griftor, E. 1994. *Mathematical Theory of Domains*. Cambridge: Cambridge University Press.
- Studeny, M. 1993. Formal Properties of Conditional Independence in Different Calculi of AI. *Pages 341–348 of*: Clarke, Michael, Kruse, Rudolf, & Moral, Serafin (eds), *Symbolic and Quantitative Approaches to Reasoning and Uncertainty*. Lecture Notes in Computer Science, vol. 747. Springer, Berlin.
- Studeny, M. 1995. Conditional Independence and Natural Conditional Functions. *Int. J. of Approximate Reasoning*, **12**(1), 43–68.